

String-orthogonal polynomials, String Equations and Two-Toda Symmetries*

M. Adler[†] P. van Moerbeke[‡]

Table of contents:

1. The 2-Toda lattice and its generic symmetries
2. A larger class of symmetries for special initial conditions
3. Borel decomposition of moment matrices, τ -functions and string-orthogonal polynomials
4. From string-orthogonal polynomials to the two-Toda lattice and the string equation
5. Virasoro constraints on two-matrix integrals.

Consider a weight $\rho(y, z)dydz = e^{V(y, z)}dydz$ on \mathbb{R}^2 , with

$$(0.1) \quad V(y, z) := V_1(y) + V_{12}(y, z) + V_2(z) := \sum_1^\infty t_i y^i + \sum_{i, j \geq 1} c_{ij} y^i z^j - \sum_1^\infty s_i z^i,$$

the corresponding inner product

$$(0.2) \quad \langle f, g \rangle = \int_{\mathbb{R}^2} dydz \rho(y, z) f(y) g(z),$$

*A final version of the present paper has appeared in: *Comm. Pure and Appl. Math.*, **50** 241–290 (1997).

[†]Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA (adler@math.brandeis.edu). The support of a National Science Foundation grant # DMS 95-4-51179 is gratefully acknowledged.

[‡]Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium (vanmoerbeke@geom.ucl.ac.be) and Brandeis University, Waltham, Mass 02254, USA (vanmoerbeke@math.brandeis.edu). The support of National Science Foundation # DMS 95-4-51179, Nato, FNRS and Francqui Foundation grants is gratefully acknowledged.

and the moment matrix,

$$(0.3) \quad m_n(t, s, c) =: (\mu_{ij})_{0 \leq i, j \leq n-1} = (\langle y^i, z^j \rangle)_{0 \leq i, j \leq n-1}.$$

As a function of t, s and c , the moment matrix $m_\infty(t, s, c)$ satisfies the differential equations

$$(0.3') \quad \frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty \Lambda^{\top n}, \quad \frac{\partial m_\infty}{\partial c_{ij}} = \Lambda^i m_\infty \Lambda^{\top j},$$

where Λ is the semi-infinite shift matrix defined below.

Consider the Borel decomposition of the semi-infinite matrix¹

$$m_\infty = S_1^{-1} S_2 \text{ with } S_1 \in \mathcal{D}_{-\infty, 0}, \quad S_2 \in \mathcal{D}_{0, \infty}$$

with S_1 having 1's on the diagonal, and S_2 having h_i 's on the diagonal, with (see section 3)

$$h_i = \frac{\det m_{i+1}}{\det m_i}.$$

The Borel decomposition $m_\infty = S_1^{-1} S_2$ above leads to two strings $(p^{(1)}(y), p^{(2)}(z))$ of monic polynomials in one variable, constructed, in terms of the character $\bar{\chi}(z) = (z^n)_{n \in \mathbf{Z}, n \geq 0}$, as follows:

$$(0.4) \quad p^{(1)}(y) =: S_1 \bar{\chi}(y) \quad p^{(2)}(z) =: h(S_2^{-1})^\top \bar{\chi}(z).$$

We call these two sequences *string-orthogonal polynomials*; indeed the Borel decomposition of $m_\infty = S_1^{-1} S_2$ above is equivalent to the orthogonality relations:

$$\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{n,m} h_n.$$

We show in section 4 the string-orthogonal polynomials have the following expressions in terms of Schur differential polynomials²

$$p_n^{(1)}(y) = \sum_{0 \leq k \leq n} \frac{p_{n-k}(-\tilde{\partial}_t) \det m_n(t, s, c)}{\det m_n(t, s, c)} y^k, \quad p_n^{(2)}(z) = \sum_{0 \leq k \leq n} \frac{p_{n-k}(\tilde{\partial}_s) \det m_n(t, s, c)}{\det m_n(t, s, c)} z^k$$

¹ $\mathcal{D}_{k,\ell}$ ($k < \ell \in \mathbf{Z}$) denotes the set of band matrices with zeros outside the strip (k, ℓ) .

²The Schur polynomials p_k , defined by $e^{\sum_{i=1}^\infty t_i z^i} = \sum_{k=0}^\infty p_k(t) z^k$ and $p_k(t) = 0$ for $k < 0$, and not to be confused with the string-orthogonal polynomials $p_i^{(k)}$, $k = 1, 2$, lead to differential polynomials

$$p_k(\pm \tilde{\partial}_t) = p_k\left(\pm \frac{\partial}{\partial t_1}, \pm \frac{1}{2} \frac{\partial}{\partial t_2}, \pm \frac{1}{3} \frac{\partial}{\partial t_3}, \dots\right)$$

To the best of our knowledge, string-orthogonal polynomials were considered for the first time, in the context of symmetric weights $\rho(y, z)dydz$, by Mehta (see [M] and [CMM]).

The *main message* of this work is to show (i) that the expressions $\det m_N$ satisfy the KP-hierarchy in t and s for each $N = 1, 2, \dots$, (section 3) (ii) that the $\det m_N$ hang together in a very specific way (they form the building blocks of the 2-Toda lattice) (section 4), (iii) that $\det m_N$ satisfies an additional Virasoro-like algebra of partial differential equations (section 5), as a result of so-called string equations (section 4). In [AvM2], we have obtained similar results for moment matrices associated with general weights on \mathbf{R} (thus connecting with the standard theory of orthogonal polynomials), rather than on \mathbf{R}^2 . It would not be difficult to generalize the results of this paper to more general weights $\rho(y, z)dydz$ on \mathbf{R}^2 , besides those of (0.1).

In terms of the matrix operators

$$\Lambda := \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ & 0 & 0 & \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad \varepsilon := \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & 0 & \\ & 2 & 0 & \\ & & & \ddots \end{pmatrix}$$

acting on $\bar{\chi}$ as

$$\Lambda \bar{\chi}(z) = z \bar{\chi}(z), \quad \varepsilon \bar{\chi}(z) = \frac{\partial}{\partial z} \bar{\chi}(z),$$

the matrices

$$L_1 := S_1 \Lambda S_1^{-1}, L_2 := S_2 \Lambda^\top S_2^{-1}, Q_1 := S_1 \varepsilon S_1^{-1}, Q_2 := S_2 \varepsilon^\top S_2^{-1}$$

interact with the vector of string-orthogonal polynomials, as follows:

$$(0.5) \quad L_1 p^{(1)}(y) = y p^{(1)}(y) \quad Q_1 p^{(1)}(y) = \frac{d}{dy} p^{(1)}(y)$$

$$h L_2^\top h^{-1} p^{(2)}(z) = z p^{(2)}(z) \quad h Q_2^\top h^{-1} p^{(2)}(z) = \frac{d}{dz} p^{(2)}(z).$$

The semi-infinite matrix L_1 (respectively L_2) is lower-triangular (resp. lower-triangular), with one subdiagonal above (resp. below); Q_1 (resp. Q_2) is strictly lower-triangular (resp. strictly upper-triangular). In Theorem 4.1 we prove the matrices L_i and Q_i satisfy so-called “*string equations*”

$$(0.6) \quad Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) = 0, \quad Q_2 + \frac{\partial V}{\partial z}(L_1, L_2) = 0.$$

When

$$V(y, z) = \sum_1^{\ell_1} t_i y^i + cyz - \sum_1^{\ell_2} s_i z^i,$$

then (0.6) implies that L_1 is a $\ell_2 + 1$ -band matrix, and L_2 is a $\ell_1 + 1$ -band matrix.

Moreover, as a function of (t, s) , the couple $L := (L_1, L_2)$ satisfies the “*two-Toda lattice equations*”, and as a function of $c_{\alpha, \beta}$, L satisfies another hierarchy of commuting vector fields; so, in terms of an appropriate Lie algebra splitting $()_+$ and $()_-$, to be explained in section 1, we have

$$(0.7) \quad \frac{\partial L}{\partial t_n} = [(L_1^n, 0)_+, L] \quad \frac{\partial L}{\partial s_n} = [(0, L_2^n)_+, L], \quad \frac{\partial L}{\partial c_{\alpha, \beta}} = -[(L_1^\alpha L_2^\beta, 0)_-, L],$$

and, what is equivalent, the moment matrix m_∞ satisfies the differential equations (0.3'), with solution (thinking of $t_n = c_{n0}$ and $s_n = -c_{0n}$):

$$m_\infty(t, s, c) = \sum_{\substack{(r_{\alpha\beta})_{\alpha, \beta \geq 0} \in \mathbf{Z}^\infty \\ (\alpha, \beta) \neq (0, 0)}} \left(\prod_{(\alpha, \beta)} \frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!} \right) \Lambda^{\sum_{\alpha \geq 1} \alpha r_{\alpha\beta}} m_\infty(0, 0, 0) \Lambda^\top \sum_{\beta \geq 1} \beta r_{\alpha\beta};$$

in particular

$$m_\infty(t, s, 0) = e^{\sum_1^\infty t_n \Lambda^n} m_\infty(0, 0, 0) e^{-\sum_1^\infty s_n \Lambda^{\top n}}.$$

Thus the integrable system under consideration in this paper is a $c_{\alpha\beta}$ -deformation of the 2-Toda lattice, which itself is an isospectral deformation of a couple of matrices (L_1, L_2) , in general bi-infinite, depending on two sequences of times t_1, t_2, \dots and s_1, s_2, \dots

Moreover, the determinant of the moment matrix has many different expressions; in particular, in terms of the moment matrix at $t = s = 0$, using the matrix $E_N(t) :=$ (the first N rows of $e^{\sum_1^\infty t_n \Lambda^n}$) of Schur polynomials $p_n(t)$ (see section 3). It can also be expressed as a 2-matrix integral reminiscent of 2-matrix integrals in string theory and in terms of the diagonal elements h_i of the upper-triangular matrix S_2 :

(0.8)

$$\begin{aligned} N! \det m_N(t, s, c) &= N! \det \left(E_N(t) m_\infty(0, 0, c) E_N(-s)^\top \right) \\ &= \int \int_{u, v \in \mathbb{R}^N} e^{\sum_{k=1}^N V(u_k, v_k)} \prod_{i < j} (u_i - u_j) \prod_{i < j} (v_i - v_j) du dv \\ &= \prod_0^{N-1} h_i(t, s, c) \\ &=: \tau_N(t, s, c). \end{aligned}$$

Finally $\tau_N(t, s, c)$ is a “ τ -function” in the sense of Sato, separately in t and s^3 .

The string equations (0.6) play an important role: they have many consequences! In Theorem 5.1, we show $\tau_N = \det m_N(t, s, c)$ satisfies the following set of constraints⁴

$$(0.9) \quad \left(J_i^{(2)} + (2N + i + 1)J_i^{(1)} + N(N + 1)J_i^{(0)} + 2 \sum_{r,s \geq 1} r c_{rs} \frac{\partial}{\partial c_{i+r,s}} \right) \tau_N = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N + i + 1)\tilde{J}_i^{(1)} + N(N + 1)\tilde{J}_i^{(0)} + 2 \sum_{r,s \geq 1} s c_{rs} \frac{\partial}{\partial c_{r,s+i}} \right) \tau_N = 0,$$

for $i \geq -1$ and $N \geq 0$.

When $V_{12} = cyz$, the relations (0.9) reduce to an inductive system of partial differential equations in t and s , for $\tau_0, \tau_1, \tau_2, \dots$, (Corollary 5.1.1)

(0.10)

$$\left(J_i^{(2)} + (2N + i + 1)J_i^{(1)} + N(N + 1)J_i^{(0)} \right) \tau_N + 2c p_{i+N}(\tilde{\partial}_t) p_N(-\tilde{\partial}_s) \tau_1 \circ \tau_{N-1} = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N + i + 1)\tilde{J}_i^{(1)} + N(N + 1)\tilde{J}_i^{(0)} \right) \tau_N + 2c p_N(\tilde{\partial}_t) p_{i+N}(-\tilde{\partial}_s) \tau_1 \circ \tau_{N-1} = 0$$

³Sato’s τ -function $\tau(t)$ in $t \in \mathbf{C}^\infty$ is the determinant of the projection $e^{-\sum t_i z^i} W \rightarrow H_+ = \{1, z, z^2, \dots\}$, where W is a fixed span of functions in z with poles at $z = \infty$ of order $k = 0, 1, 2, \dots$. Equivalently, a τ -function satisfies the bilinear relations

$$\oint_C \tau(t - [z^{-1}]) \tau(t' + [z^{-1}]) e^{\sum_{i=1}^\infty (t_i - t'_i) z^i} dz = 0,$$

where C is a small contour about $z = \infty$ and where $[\alpha] := \left(\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots \right) \in \mathbf{C}^\infty$. The bilinear relations imply that $\tau(t)$ satisfies the KP-hierarchy.

⁴in terms of the customary Virasoro generators in t_1, t_2, \dots :

$$J_n^{(0)} = \delta_{n0}, \quad J_n^{(1)} = \frac{\partial}{\partial t_n} + (-n)t_{-n}, \quad J_0^{(1)} = 0$$

$$J_n^{(2)} = \sum_{i+j=n} : J_i^{(1)} J_j^{(1)} : = \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{-i+j=n} i t_i \frac{\partial}{\partial t_j} + \sum_{-i-j=n} (i t_i)(j t_j),$$

where “ $:$ ” denotes normal ordering, i.e., always pull differentiation to the right, and where a symbol $= 0$, whenever it does not make sense; e.g., $\partial/\partial t_n = 0$ for $n \leq 0$. We also define Virasoro generators in s , namely $\tilde{J}_n^{(k)} := J_n^{(k)} \Big|_{t \rightarrow s}$.

for $i \geq -1$ and $N \geq 1$.
involving the Hirota operation⁵; to be precise, the expression $p_n(\tilde{\partial}_t)p_m(-\tilde{\partial}_s)f \circ g$, for $\tilde{\partial}_t$ and $\tilde{\partial}_s$ spelled out in footnote 2, is defined as the coefficient in the (u, v) -Taylor expansion of

$$\begin{aligned} f(t + [u], s - [v])g(t - [u], s + [v]) &= \sum_{n,m \geq 0} u^n v^m p_n(\tilde{\partial}_t)p_m(-\tilde{\partial}_s)f \circ g \\ &= \sum_{n,m \geq 0} u^n v^m \sum_{\substack{i+i'=n \\ j+j'=m \\ i,i',j,j' \geq 0}} p_i(\tilde{\partial}_t)p_j(-\tilde{\partial}_s)f \cdot p_{i'}(-\tilde{\partial}_t)p_{j'}(\tilde{\partial}_s)g. \end{aligned}$$

Also, in terms of W-generators, to be defined in (1.36), the matrix integral (0.8) satisfies (Theorem 5.2):

$$(0.11) \quad \left(\sum_{k=0}^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{N,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{N-1,j-i}^{(i-k+1)} \right) \tau_N = \frac{i!}{(-c)^i} \delta_{ij} \tau_N, \quad i, j \geq 0,$$

where $\bar{\alpha}_k^{(i,j)}$ and $\bar{\beta}_k^{(i)}$ are numbers defined in (5.10). A relation of this type was conjectured by Morozov [M].

The technique employed here is the one of symmetry vector fields of integrable systems (KP, Toda, two-Toda, etc...), as developped in [ASV1] and [AV2]; they are non-local and time-dependent vector fields transversal to and commuting with the integrable hierarchy, when acting on the (*explicitly*) time-dependent wave functions; however bracketing a symmetry with a Toda vector field (0.7), which acts on the (*implicitly*) time-dependent L , yields another vector field in the hierarchy. In [ASV], we have shown that for generic initial conditions the 2-Toda lattice admits a $w_\infty \times w_\infty$ algebra of symmetries, i.e., an algebra without central extension.

In this study, we consider special initial conditions, preserved by the Toda flows; we show they admit a much wider algebra of symmetries, to be described in Theorem 2.1. This big algebra contains not only the algebra $w_\infty \times w_\infty$ above, but also a Kac-Moody extension $w_\infty \otimes \mathbf{C}(h, h^{-1})$ of w_∞ (in two different ways). Via the Adler-Shiota-van Moerbeke formula, symmetries at the level of the wave functions induce

⁵Given two differential polynomials $p(\partial_t)$ and $q(\partial_s)$, and functions $f(t_1, t_2, \dots; s_1, s_2, \dots)$, $g(t_1, t_2, \dots; s_1, s_2, \dots)$, define the Hirota operation for shifts $y = (y_1, y_2, \dots)$ and $z = (z_1, z_2, \dots)$:

$$p(\partial_t)q(\partial_s)f \circ g(t, s) := p\left(\frac{\partial}{\partial y}\right)q\left(\frac{\partial}{\partial z}\right)f(t + y, s + z)g(t - y, s - z)\Big|_{y=z=0}.$$

symmetries at the level of the τ -function; these symmetries form an algebra with central extension. van de Leur [vdL1,2] has a striking representation-theoretical generalization of the ASV-formula; it is an open question to understand whether it includes the wider class of symmetries under consideration here.

The string equations (0.6) are equivalent to the vanishing of a whole algebra of symmetries, viewed as vector fields on the manifold of wave functions; they lead, upon using the ASV-formula, to the algebra of constraints (0.9) and (0.10) (on the 2-matrix integral), viewed as vector fields on the manifold of τ -functions, and this in a straightforward, conceptual and precise fashion.

We wish to thank S. D'Addato for a superb typing job.

1 The 2-Toda lattice and its generic symmetries

Define the column vector $\chi(z) = (z^n)_{n \in \mathbf{Z}}$, and matrix operators $\Lambda, \Lambda^*, \varepsilon, \varepsilon^*$ defined as follows:

$$\begin{aligned}\Lambda\chi(z) &= z\chi(z), \quad \varepsilon\chi(z) = \frac{\partial}{\partial z}\chi(z), \\ \Lambda^*\chi(z) &= z^{-1}\chi(z), \quad \varepsilon^*\chi(z) = \frac{\partial}{\partial z^{-1}}\chi(z).\end{aligned}$$

Note that

$$\Lambda^* = \Lambda^\top = \Lambda^{-1}, \quad \varepsilon^* = -\varepsilon^\top + \Lambda,$$

and

$$(1.1) \quad \begin{cases} \Lambda^\top\chi(z^{-1}) = z\chi(z^{-1}), \quad \Lambda\chi(z^{-1}) = z^{-1}\chi(z^{-1}), \\ \varepsilon^\top\chi(z^{-1}) = z^{-1}\chi(z^{-1}) - \frac{\partial}{\partial z}\chi(z^{-1}) \\ \varepsilon^{*\top}\chi(z^{-1}) = z\chi(z^{-1}) - \frac{\partial}{\partial z^{-1}}\chi(z^{-1}). \end{cases}$$

Consider the splitting of the algebra \mathcal{D} of pairs (P_1, P_2) of infinite $(\mathbf{Z} \times \mathbf{Z})$ matrices such that $(P_1)_{ij} = 0$ for $j - i \gg 0$ and $(P_2)_{ij} = 0$ for $i - j \gg 0$, used in [ASV2]; to wit:

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_+ + \mathcal{D}_-, \\ \mathcal{D}_+ &= \{(P, P) \mid P_{ij} = 0 \text{ if } |i - j| \gg 0\} = \{(P_1, P_2) \in \mathcal{D} \mid P_1 = P_2\}, \\ \mathcal{D}_- &= \{(P_1, P_2) \mid (P_1)_{ij} = 0 \text{ if } j \geq i, (P_2)_{ij} = 0 \text{ if } i > j\},\end{aligned}$$

with $(P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-$ given by

$$(1.2) \quad \begin{aligned} (P_1, P_2)_+ &= (P_{1u} + P_{2\ell}, P_{1u} + P_{2\ell}), \\ (P_1, P_2)_- &= (P_{1\ell} - P_{2\ell}, P_{2u} - P_{1u}); \end{aligned}$$

P_u and P_ℓ denote the upper (including diagonal) and strictly lower triangular parts of the matrix P , respectively.

The two-dimensional Toda lattice equations

$$(1.3) \quad \frac{\partial L}{\partial t_n} = [(L_1^n, 0)_+, L] \quad \text{and} \quad \frac{\partial L}{\partial s_n} = [(0, L_2^n)_+, L] \quad n = 1, 2, \dots$$

are deformations of a pair of infinite matrices

$$(1.4) \quad L = (L_1, L_2) = \left(\sum_{-\infty < i \leq 1} a_i^{(1)} \Lambda^i, \sum_{-1 \leq i < \infty} a_i^{(2)} \Lambda^i \right) \in \mathcal{D},$$

where $\Lambda = (\delta_{j-i,1})_{i,j \in \mathbf{Z}}$, and $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $t = (t_1, t_2, \dots)$ and $s = (s_1, s_2, \dots)$, such that

$$a_1^{(1)} = I \quad \text{and} \quad (a_{-1}^{(2)})_{nn} \neq 0 \quad \text{for all } n.$$

In analogy with Sato's theory, Ueno and Takasaki [U-T] show a solution L of (1.3) has the representation

$$L_1 = W_1 \Lambda W_1^{-1} = S_1 \Lambda S_1^{-1}, \quad L_2 = W_2 \Lambda^{-1} W_2^{-1} = S_2 \Lambda^{-1} S_2^{-1}$$

in terms of two pairs of wave operators

$$\begin{cases} S_1 = \sum_{i \leq 0} c_i(t, s) \Lambda^i, & S_2 = \sum_{i \geq 0} c'_i(t, s) \Lambda^i \\ c_i, c'_i : \text{diagonal matrices, } c_0 = I, (c'_0)_{ii} \neq 0, & \text{for all } i \end{cases}$$

and

$$(1.5) \quad W_1 = S_1(t, s) e^{\sum_1^\infty t_k \Lambda^k}, \quad W_2 = S_2(t, s) e^{\sum_1^\infty s_k \Lambda^{-k}}.$$

One also introduces a pair of wave vectors $\Psi = (\Psi_1, \Psi_2)$, and a pair of adjoint wave vectors, $\Psi^* = (\Psi_1^*, \Psi_2^*)$, instead of a single wave function and a single adjoint wave function:

$$(1.6) \quad \begin{cases} \Psi_1(t, s, z) = W_1 \chi(z) = e^{\sum_1^\infty t_k z^k} S_1 \chi(z) \\ \Psi_2(t, s, z) = W_2 \chi(z) = e^{\sum_1^\infty s_k z^{-k}} S_2 \chi(z) \end{cases}$$

and

$$(1.7) \quad \begin{cases} \Psi_1^* = (W_1^\top)^{-1} \chi^*(z) = e^{-\sum_1^\infty t_k z^k} (S_1^\top)^{-1} \chi(z^{-1}) \\ \Psi_2^* = (W_2^\top)^{-1} \chi^*(z) = e^{-\sum_1^\infty s_k z^{-k}} (S_2^\top)^{-1} \chi(z^{-1}). \end{cases}$$

The wave functions Ψ and Ψ^* evolve in t and s according to the following differential equations⁶:

$$(1.8) \quad \begin{cases} \frac{\partial \Psi}{\partial t_n} = (L_1^n, 0)_+ \Psi = ((L_1^n)_u, (L_1^n)_u) \Psi \\ \frac{\partial \Psi}{\partial s_n} = (0, L_2^n)_+ \Psi = ((L_2^n)_\ell, (L_2^n)_\ell) \Psi \end{cases}$$

$$(1.9) \quad \begin{cases} \frac{\partial}{\partial t_n} \Psi^* = ((L_1^n, 0)_+)^{\top} \Psi^* \\ \frac{\partial}{\partial s_n} \Psi^* = -((0, L_2^n)_+)^{\top} \Psi^*. \end{cases}$$

Besides $L = (L_1, L_2) = (S_1 \Lambda S_1^{-1}, S_2 \Lambda^\top S_2^{-1})$, the operators

$$L^* = (L_1^*, L_2^*) = (L_1^\top, L_2^\top)$$

$$M = (M_1, M_2) := (W_1 \varepsilon W_1^{-1}, W_2 \varepsilon^* W_2^{-1}) = \left(S_1 \left(\varepsilon + \sum_1^\infty k t_k \Lambda^{k-1} \right) S_1^{-1}, S_2 \left(\varepsilon^* + \sum_1^\infty k s_k \Lambda^{\top k-1} \right) S_2^{-1} \right)$$

$$M^* = (M_1^*, M_2^*) = (-M_1^\top + L_1^{\top-1}, -M_2^\top + L_2^{\top-1})$$

satisfy, in view of (1.1):

$$(1.10) \quad \begin{aligned} L\Psi &= (z, z^{-1})\Psi, & M\Psi &= \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial(z^{-1})} \right) \Psi, & [L, M] &= (1, 1), \\ L^*\Psi^* &= (z, z^{-1})\Psi^*, & M^*\Psi^* &= \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial(z^{-1})} \right) \Psi^*, & [L^*, M^*] &= (1, 1). \end{aligned}$$

The operators L, M, L^\top, M^\top and $W := (W_1, W_2)$ evolve according to

$$(1.11) \quad \begin{cases} \frac{\partial}{\partial t_n} \begin{pmatrix} L \\ M \end{pmatrix} = \left[(L_1^n, 0)_+, \begin{pmatrix} L \\ M \end{pmatrix} \right] \\ \frac{\partial}{\partial s_n} \begin{pmatrix} L \\ M \end{pmatrix} = \left[(0, L_2^n)_+, \begin{pmatrix} L \\ M \end{pmatrix} \right] \end{cases}$$

⁶Here the action is viewed componentwise, e.g., $(A, B)\Psi = (A\Psi_1, B\Psi_2)$ or $(z, z^{-1})\Psi = (z\Psi_1, z^{-1}\Psi_2)$.

$$(1.12) \quad \begin{cases} \frac{\partial}{\partial t_n} \begin{pmatrix} L^\top \\ M^\top \end{pmatrix} = \left[-((L_1^n, 0)_+)^{\top}, \begin{pmatrix} L^\top \\ M^\top \end{pmatrix} \right] \\ \frac{\partial}{\partial s_n} \begin{pmatrix} L^\top \\ M^\top \end{pmatrix} = \left[-(0, L_2^n)_+)^{\top}, \begin{pmatrix} L^\top \\ M^\top \end{pmatrix} \right] \end{cases}$$

$$(1.13) \quad \begin{cases} \frac{\partial W}{\partial t_n} = (L_1^n, 0)_+ W, \\ \frac{\partial W}{\partial s_n} = (0, L_2^n)_+ W \end{cases}$$

and consequently

$$\begin{aligned} \frac{\partial W_1^{-1} W_2}{\partial t_n} &= -W_1^{-1} (L_1^n)_u W_1 W_1^{-1} W_2 + W_1^{-1} (L_1^n)_u W_2 = 0 \\ \frac{\partial W_1^{-1} W_2}{\partial s_n} &= 0. \end{aligned}$$

This implies that

$$(1.14) \quad W_1^{-1} W_2(t, s) = W_1^{-1} W_2(0, 0)$$

and thus

$$(1.15) \quad (S_1^{-1} S_2)(t, s) = e^{\Sigma t_n \Lambda^n} (S_1^{-1} S_2)(0, 0) e^{-\Sigma s_n (\Lambda^\top)^n}.$$

Ueno and Takasaki [U-T] show that the 2-Toda deformations of Ψ , and hence L , can ultimately all be expressed in terms of one sequence of τ -functions

$$\tau(n, t, s) = \tau_n(t_1, t_2, \dots; s_1, s_2, \dots) = \det[(S_1^{-1} S_2(t, s))_{i,j}]_{-\infty \leq i, j \leq n-1}, \quad n \in \mathbf{Z} :$$

to wit:

$$(1.16) \quad \Psi_1(t, s, z) = \left(\frac{e^{-\eta} \tau_n(t, s)}{\tau_n(t, s)} e^{\sum t_i z^i} z^n \right)_{n \in \mathbf{Z}}, \quad \Psi_2(t, s, z) = \left(\frac{e^{-\tilde{\eta}} \tau_{n+1}(t, s)}{\tau_n(t, s)} e^{\sum s_i z^{-i}} z^n \right)_{n \in \mathbf{Z}}$$

$$(1.17) \quad \Psi_1^*(t, s, z) = \left(\frac{e^{\eta} \tau_{n+1}(t, s)}{\tau_{n+1}(t, s)} e^{-\sum t_i z^i} z^{-n} \right)_{n \in \mathbf{Z}}, \quad \Psi_2^*(t, s, z) = \left(\frac{e^{\tilde{\eta}} \tau_n(t, s)}{\tau_{n+1}(t, s)} e^{-\sum s_i z^{-i}} z^{-n} \right)_{n \in \mathbf{Z}}.$$

where

$$(1.18) \quad \eta = \sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i} \quad \text{and} \quad \tilde{\eta} = \sum_1^\infty \frac{z^i}{i} \frac{\partial}{\partial s_i},$$

so that

$$(1.19) \quad e^{a\eta+b\tilde{\eta}} f(t, s) = f(t + a[z^{-1}], s + b[z])$$

with $[\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, \dots)$. Here the labeling of Ψ_i^* is slightly different from the one of [U-T].

The symmetries of the 2-Toda hierarchy are conveniently expressed in terms of the operators L and M . In view of the relation

$$(1.20) \quad z^\alpha \left(\frac{\partial}{\partial z} \right)^\beta \Psi_1 = M_1^\beta L_1^\alpha \Psi_1, \quad u^\alpha \left(\frac{\partial}{\partial u} \right)^\beta \Big|_{u=z^{-1}} \Psi_2 = M_2^\beta L_2^\alpha \Psi_2,$$

the Lie algebra

$$w_\infty := \text{span} \left\{ z^\alpha \left(\frac{\partial}{\partial z} \right)^\beta \mid \alpha, \beta \in \mathbf{Z}, \beta \geq 0 \right\}$$

comes naturally into play. To be precise, denoting by ϕ the algebra antihomomorphism

$$\phi: w_\infty \times w_\infty \rightarrow \mathcal{D} : \begin{cases} (z^\alpha (\partial/\partial z)^\beta, 0) \mapsto (M_1^\beta L_1^\alpha, 0) \\ (0, u^\alpha (\partial/\partial u)^\beta) \mapsto (0, M_2^\beta L_2^\alpha) \end{cases}$$

we have a Lie algebra antihomomorphism⁷

$$\begin{aligned} \mathbf{Y}: w_\infty \times w_\infty &\rightarrow \{\text{symmetries on } \Psi, \Psi^*, L, L^\top, M, M^\top\} \\ p &\longmapsto \begin{cases} \mathbf{Y}_p \Psi = -\phi(p)_- \Psi, \quad \mathbf{Y}_p \Psi^* = (\phi(p)_-)^\top \Psi^*, \\ \mathbf{Y}_p L = [-\phi(p)_-, L], \quad \mathbf{Y}_p L^\top = [(\phi(p)_-)^\top, L], \\ \mathbf{Y}_p M = [-\phi(p)_-, M], \quad \mathbf{Y}_p M^\top = [(\phi(p)_-)^\top, M^\top]. \end{cases} \end{aligned}$$

So for any admissible⁸ polynomial q of L and M , we have

$$(1.21) \quad \mathbf{Y}_p(q) = [-p_-, q] \quad \text{and} \quad \mathbf{Y}_p \Psi = -p_- \Psi.$$

⁷Thus $-\mathbf{Y}$ is a Lie algebra homomorphism; note we shall more often denote \mathbf{Y}_p by $\mathbf{Y}_{\phi(p)}$.

⁸The products $M_i^\beta L_i^\alpha$ are admissible, but, in this generality, the product $L_1 L_2$ obviously does not make sense.

To prove

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}] = -\mathbf{Y}_{[p_1, p_2]},$$

one computes

$$[\mathbf{Y}_{p_1}, \mathbf{Y}_{p_2}]\Psi = -Z_1\Psi,$$

where, upon using \mathcal{D}_- and \mathcal{D}_+ are Lie subalgebras,

$$\begin{aligned} Z_1 &:= -(\mathbf{Y}_{p_1}(p_2))_- + (\mathbf{Y}_{p_2}(p_1))_- - [p_{1-}, p_{2-}] \\ &= [p_{1-}, p_2]_- + [-p_{2-}, p_1]_- - [p_{1-}, p_{2-}] = [p_1, p_2]_-. \end{aligned}$$

Moreover

$$[\mathbf{Y}_p, \frac{\partial}{\partial t_n}]\Psi = [\mathbf{Y}_p, \frac{\partial}{\partial s_n}]\Psi = 0;$$

indeed, from (1.8), the vector field,

$$(1.22) \quad \mathbf{Z}_q\Psi = q_+\Psi, \quad \mathbf{Z}_qp = [q_+, p]$$

represents $\frac{\partial}{\partial t_n}$ for $q = (L_1^n, 0)$ and $\frac{\partial}{\partial s_n}$ for $q = (0, L_2^n)$; now one checks

$$(1.23) \quad [\mathbf{Y}_p, \mathbf{Z}_q]\Psi = -Z_2\Psi$$

where

$$Z_2 = (\mathbf{Z}_q(p))_- + (\mathbf{Y}_p(q))_+ - [q_+, p_-] = [q_+, p]_- + [-p_-, q]_+ - [q_+, p_-] = 0.$$

Remark 1: Some explanation is necessary to understand the equation above $\mathbf{Y}_p\Psi^* = (\phi(p)_-)^{\top}\Psi^*$: the flows on the transposed matrices L^{\top}, M^{\top} are immediately obtained from the original flows by just taking transposes; the flows on the adjoint wave operator Ψ^* are obtained as follows: since

$$(1.24) \quad \begin{cases} (\Psi_1, \Psi_2) = (W_1, W_2)\chi(z) \\ (\Psi_1^*, \Psi_2^*) = ((W_1^{-1})^{\top}, (W_2^{-1})^{\top})\chi(z^{-1}), \end{cases}$$

we have for any deformation $'$, that

$$\Psi_i' = W_i'\chi(z), \quad \Psi_i^{*'} = (W_i^{-1})^{\top'}\chi(z).$$

Thus, we have the straightforward equivalences:

$$W_i' = A_i W_i \iff \Psi_i' = A \Psi_i$$

and

$$(W_i^{-1})^{\top'} = -A^\top (W_i^{-1})^\top \iff \Psi_i^{*'} = -A^\top \Psi_i^*;$$

so, we conclude

$$(1.25) \quad \Psi_i' = A\Psi_i \iff \Psi_i^{*'} = -A^\top \Psi_i^*.$$

Remark 2: Spelling out the symmetries (1.21) on L and Ψ , one finds

$$(1.26) \quad \mathbf{Y}_{M_i^\alpha L_i^\beta} L = (-1)^{i-1} \left[\left(-(M_i^\alpha L_i^\beta)_\ell, (M_i^\alpha L_i^\beta)_u \right), L \right]$$

$$(1.27) \quad \mathbf{Y}_{M_i^\alpha L_i^\beta} \Psi = (-1)^{i-1} \left(-(M_i^\alpha L_i^\beta)_\ell, (M_i^\alpha L_i^\beta)_u \right) \Psi,$$

and the equivalent equations, upon taking the transpose of the previous ones,

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} L^\top = (-1)^{i-1} \left[\left((M_i^\alpha L_i^\beta)_\ell, -(M_i^\alpha L_i^\beta)_u \right)^\top, L^\top \right]$$

(1.27')

$$\mathbf{Y}_{M_i^\alpha L_i^\beta} \Psi^* = (-1)^{i-1} \left((M_i^\alpha L_i^\beta)_\ell, -(M_i^\alpha L_i^\beta)_u \right)^\top \Psi^*.$$

The action of a vector field on Ψ , commuting with the Toda flows, induces an action on τ via (1.16), thus leading to the following relation between logarithmic derivatives of Ψ and τ :

$$(1.28) \quad \left(\frac{\Psi_1'}{\Psi_1} \right)_n = (e^{-\eta} - 1) \frac{\tau_n'}{\tau_n}, \quad \left(\frac{\Psi_2'}{\Psi_2} \right)_n = (e^{-\tilde{\eta}} - 1) \frac{\tau_{n+1}'}{\tau_{n+1}} + \left(\frac{\tau_{n+1}'}{\tau_{n+1}} - \frac{\tau_n'}{\tau_n} \right).$$

Proposition 1.1. *Given the symmetry vector field \mathbf{Y}_p , the following relation holds:*

$$(1.29) \quad \mathbf{Y}_p \log \frac{\tau_{n+1}}{\tau_n} = p_{nn}$$

Proof: On the one hand, taking into account the expression (1.16) for Ψ_2 , we have, in view of the fact that $e^{-\tilde{\eta}} f(t, s) = f(t, s - [z])$, the following Taylor expansion

about $z = 0$:

$$\begin{aligned}
\left(\frac{\mathbf{Y}_p \Psi_2}{\Psi_2}\right)_n &= \left(\frac{p_u \Psi_2}{\Psi_2}\right)_n \\
\text{(a)} \quad &= \frac{(p_{nn} + O(z))\left(\frac{\tau_{n+1}(t,s)}{\tau_n(t,s)} + O(z)\right)z^n}{\left(\frac{\tau_{n+1}(t,s)}{\tau_n(t,s)} + O(z)\right)z^n} \\
&= p_{nn} + O(z).
\end{aligned}$$

On the other hand,

$$\text{(b)} \quad (e^{-\tilde{\eta}} - 1) \frac{\tau'_{n+1}}{\tau_{n+1}} + \left(\frac{\tau'_{n+1}}{\tau_{n+1}} - \frac{\tau'_n}{\tau_n}\right) = \frac{\tau'_{n+1}}{\tau_{n+1}} - \frac{\tau'_n}{\tau_n} + O(z) = \left(\log \frac{\tau_{n+1}}{\tau_n}\right)' + O(z).$$

According to (1.28), the expressions (a) and (b) are equal, leading to (1.29).

In order to spell out the precise relationship between the symmetry vector fields on Ψ , and those acting on τ , we need to discuss generating functions for the symmetries.

A generating function of the **symmetries on the Ψ -manifold** is given by

$$(1.30) \quad \mathbf{Y}_N \Psi = -N_- \Psi$$

which naturally leads to the symmetry on the L -manifold expressed in the Lax form

$$(1.31) \quad \mathbf{Y}_N L = [-N_-, L],$$

with

$$(1.32) \quad N = (N_1, 0) \quad \text{or} \quad (0, N_2),$$

$$N_i := (\mu - \lambda) e^{(\mu - \lambda) M_i} \delta(\lambda, L_i) = \sum_{k=1}^{\infty} \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^{\infty} \lambda^{-\ell-k} k(M_i^{k-1} L_i^{k-1+\ell}),$$

where $\delta(\lambda, z) := \sum_{n=-\infty}^{\infty} \lambda^{-n} z^{n-1}$.

A generating function of the **symmetries on the τ -manifold** is given by a vector of the vertex operators, based on the one of Date, Jimbo, Kashiwara and Miwa, acting on a single τ -function,

$$\begin{aligned}
(1.33) \quad X(t, \lambda, \mu) &:= \exp\left(\sum_1^\infty t_i(\mu^i - \lambda^i)\right) \exp\left(\sum_1^\infty (\lambda^{-i} - \mu^{-i}) \frac{1}{i} \frac{\partial}{\partial t_i}\right), \\
&= \sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^\infty \lambda^{-\ell-k} W_\ell^{(k)}, \quad \text{with } W_\ell^{(0)} = \delta_{\ell 0}.
\end{aligned}$$

So, the τ -manifold symmetries for the 2-Toda lattice can be expressed as a vector of vertex operators $X(t, \lambda, \mu)$ acting on the vector of τ -functions⁹:

$$(1.34) \quad \mathbf{X}(t, \lambda, \mu) := \left(\left(\frac{\mu}{\lambda} \right)^n X(t, \lambda, \mu) \right)_{n \in \mathbf{Z}} = \left(\sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^\infty \lambda^{-\ell-k} W_{n,\ell}^{(k)} \right)_{n \in \mathbf{Z}}$$

$$(1.35) \quad \tilde{\mathbf{X}}(s, \lambda, \mu) := \left(\left(\frac{\lambda}{\mu} \right)^n X(s, \lambda, \mu) \right)_{n \in \mathbf{Z}} = \left(\sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^\infty \lambda^{-\ell-k} \tilde{W}_{n,\ell}^{(k)} \right)_{n \in \mathbf{Z}}$$

with

$$(1.36) \quad W_{n,\ell}^{(k)} = \sum_{j=0}^k \binom{n}{j} (k)_j W_\ell^{(k-j)} \quad \text{and} \quad \tilde{W}_{n,\ell}^{(k)} = W_{-n,\ell}^{(k)} \Big|_{t \rightarrow s}.$$

One easily computes (remember the definition of the customary Virasoro generators J 's in footnote 2)

$$(1.37) \quad W_n^{(0)} = \delta_{n,0}, \quad W_n^{(1)} = J_n^{(1)} \quad \text{and} \quad W_n^{(2)} = J_n^{(2)} - (n+1)J_n^{(1)}, \quad n \in \mathbf{Z}$$

and

$$\begin{aligned}
(1.38) \quad W_{m,i}^{(1)} &= W_i^{(1)} + mW_i^{(0)} & W_{m,i}^{(2)} &= W_i^{(2)} + 2mW_i^{(1)} + m(m-1)W_i^{(0)} \\
&= J_i^{(1)} + m\delta_{i0} & &= J_i^{(2)} + (2m-i-1)J_i^{(1)} + m(m-1)\delta_{i0}.
\end{aligned}$$

For future use, record the formulas

$$\begin{aligned}
(1.39) \quad \frac{1}{2}W_{m,i}^{(2)} + (i+1)W_{m,i}^{(1)} &= \frac{1}{2}J_i^{(2)} + \left(m + \frac{i+1}{2}\right)J_i^{(1)} + \frac{m(m+1)}{2}\delta_{i0} \\
\frac{1}{2}W_{m,i}^{(2)} - W_{m,i}^{(1)} &= \frac{1}{2}J_i^{(2)} + \left(m - \frac{i+3}{2}\right)J_i^{(1)} + \frac{m(m-3)}{2}\delta_{i0}.
\end{aligned}$$

⁹using $\left(\frac{\mu}{\lambda}\right)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} \left(\frac{\mu - \lambda}{\lambda}\right)^k$ and $\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$

The corresponding expression $\tilde{W}_{m,i}^{(k)}$ can be read off from the above, using (1.36), with $J_n^{(k)}$ replaced by $\tilde{J}_n^{(k)} = J_n^{(k)}|_{t \rightarrow s}$.

The symmetry \mathbf{Y}_N acting on Ψ and L as in (1.30) and (1.31) lifts to the vertex operator \mathbf{X} acting on τ , as in (1.34) and (1.35), according to the Adler-Shiota-van Moerbeke formula [ASV2]:

Theorem 1.2. *The vector fields \mathbf{Y}_{N_1} and \mathbf{Y}_{N_2} acting on the Ψ -manifold and the vertex operators of type $\mathbf{X}(t, \lambda, \mu)$ and $\frac{\mu}{\lambda} \tilde{\mathbf{X}}(s, \lambda, \mu)$ acting on the τ -manifold are related as follows:*

$$\begin{aligned} \frac{\mathbf{Y}_{(N_1,0)} \Psi}{\Psi} &= \left((e^{-\eta} - 1) \frac{\mathbf{X}(t, \lambda, \mu) \tau}{\tau}, (\Lambda e^{-\tilde{\eta}} - 1) \frac{\mathbf{X}(t, \lambda, \mu) \tau}{\tau} \right), \\ \frac{\mathbf{Y}_{(0,N_2)} \Psi}{\Psi} &= \frac{\mu}{\lambda} \left((e^{-\eta} - 1) \frac{\tilde{\mathbf{X}}(s, \lambda, \mu) \tau}{\tau}, (\Lambda e^{-\tilde{\eta}} - 1) \frac{\tilde{\mathbf{X}}(s, \lambda, \mu) \tau}{\tau} \right). \end{aligned}$$

Expanding \mathbf{X} and $\tilde{\mathbf{X}}$ in terms of W -generators, and N_i in terms of $M_i^\alpha L_i^\beta$, we obtain:

Corollary 1.2.1. *For $n, k \in \mathbf{Z}$, $n \geq 0$, the symmetry vector fields $\mathbf{Y}_{M_i^n L_i^{n+k}}$, ($i = 1, 2$) acting on Ψ lead to the correspondences*

$$(1.40) \quad -\frac{((M_1^n L_1^{n+k})_\ell \Psi_1)_m}{\Psi_{1,m} z^m} = \frac{1}{n+1} (e^{-\eta} - 1) \frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m},$$

$$(1.41) \quad \frac{((M_1^n L_1^{n+k})_u \Psi_2)_m}{\Psi_{2,m} z^m} = \frac{1}{n+1} \left(e^{-\tilde{\eta}} \frac{W_{m+1,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m} \right),$$

$$(1.42) \quad \frac{((M_2^n L_2^{n+k})_\ell \Psi_1)_m}{\Psi_{1,m} z^m} = \frac{1}{n+1} (e^{-\eta} - 1) \frac{\tilde{W}_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m},$$

$$(1.43) \quad -\frac{((M_2^n L_2^{n+k})_u \Psi_2)_m}{\Psi_{2,m} z^m} = \frac{1}{n+1} \left(e^{-\tilde{\eta}} \frac{\tilde{W}_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{\tilde{W}_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m} \right),$$

and acting on Ψ^*

$$(1.44) \quad \frac{(((M_1^n L_1^{n+k})_\ell)^\top \Psi_1^*)_m}{\Psi_{1,m}^* z^{-m}} = \frac{1}{n+1} (e^\eta - 1) \frac{W_{m+1,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}}$$

$$(1.45) \quad -\frac{(((M_1^n L_1^{n+k})_u)^\top \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} = \frac{1}{n+1} \left(e^{\tilde{\eta}} \frac{W_{m,k}^{(n+1)}(\tau_m)}{\tau_m} - \frac{W_{m+1,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} \right)$$

$$(1.46) \quad -\frac{(((M_2^n L_2^{n+k})_\ell)^\top \Psi_1^*)_m}{\Psi_{1,m}^* z^{-m}} = \frac{1}{n+1} (e^\eta - 1) \frac{\tilde{W}_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}}$$

$$(1.47) \quad \frac{(((M_2^n L_2^{n+k})_u)^\top \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} = \frac{1}{n+1} \left(e^{\tilde{\eta}} \frac{\tilde{W}_{m-1,k}^{(n+1)}(\tau_m)}{\tau_m} - \frac{\tilde{W}_{m,k}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} \right).$$

Remark: The expansion of the vertex operator (1.33) is obtained by multiplying the series

$$e^{\sum_1^\infty t_i (\mu^i - \lambda^i)} = \sum_{k=0}^\infty (\mu - \lambda)^k \sum_{i=1}^\infty f_i^{(k)}(t) \lambda^{i-1},$$

and

$$e^{\sum_1^\infty (\lambda^{-i} - \mu^{-i}) \frac{1}{i} \frac{\partial}{\partial t_i}} = \sum_{k=0}^\infty (\mu - \lambda)^k \sum_{i=0}^\infty g_i^{(k)} \left(\frac{\partial}{\partial t} \right) \lambda^{-k-i},$$

where $f_i^{(k)}(u)$ and $g_i^{(k)}(u)$ are polynomials in u , vanishing at $u = 0$, except $f_1^{(0)} = g_0^{(0)} = 1$. Therefore

$$X(t, \lambda, \mu) = \sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \left(\sum_{\ell=-k}^{-\infty} \lambda^{-k-\ell} k! f_{-k-\ell+1}^{(k)} + \sum_{\ell=-\infty}^\infty \lambda^{-k-\ell} (\text{differential operator}) \right),$$

showing, for all $\ell \in \mathbf{Z}$,

$$(1.48) \quad W_\ell^{(k)} = (\text{polynomial in } t, \text{ without constant term}) + \left\{ \begin{array}{c} \text{pure differential} \\ \text{operator} \end{array} \right\} + \delta_{\ell,0} \delta_{k,0},$$

with

$$W_\ell^{(k)} = (\text{pure differential operator}) \text{ if and only if } \ell \geq -k + 1 \text{ with } \ell, k \neq 0.$$

Observe, in view of (1.36), that for $n, \ell \in \mathbf{Z}$,

$$(1.49) \quad \begin{aligned} W_{n,\ell}^{(k)} &= (\text{differential operator}) + (\text{polynomial in } t, \text{ without constant term}) + (n)_k \delta_{\ell,0} \\ \tilde{W}_{n,\ell}^{(k)} &= (\text{differential operator}) + (\text{polynomial in } s, \text{ without constant term}) + (-n)_k \delta_{\ell,0}. \end{aligned}$$

2 A larger class of symmetries for special initial conditions

In this section we consider the submanifold of L , M and Ψ such that polynomials $p(L_1, M_1, L_2, M_2)$ in L_1, M_1, L_2, M_2 are legitimate objects; e.g., $L_1 L_2$ makes sense. This point of view leads to a much wider algebra of symmetries containing the ones of section 1. Henceforth, we change the point of view; namely, p refers to a polynomial and not to a vector of polynomials as in previous section.

From (1.11), it follows that, for such polynomials p , the Toda vector fields can be rewritten as:

$$(2.1) \quad \frac{\partial p}{\partial t_n} = [(L_1^n)_u, p] \quad \text{and} \quad \frac{\partial p}{\partial s_n} = [(L_2^n)_\ell, p] \quad n = 1, 2, \dots$$

Letting $p = p(L_1, M_1, L_2, M_2)$ be a polynomial, we redefine symmetries as follows

$$(2.2) \quad \mathbf{Y}_p \Psi = (-p_\ell \Psi_1, p_u \Psi_2) = \begin{cases} -(p, 0)_- \Psi \\ (0, p)_- \Psi \end{cases}$$

and thus

$$(2.3) \quad \mathbf{Y}_p(L_1, L_2) = ([-p_\ell, L_1], [p_u, L_2]) = \begin{cases} [-(p, 0)_-, L] \\ [(0, p)_-, L] \end{cases}$$

and similarly for L_1, L_2 replaced by M_1, M_2 . Then

$$(2.4) \quad \begin{aligned} [\mathbf{Y}_p, \mathbf{Y}_{\bar{p}}] \Psi &= \left((\mathbf{Y}_p(\bar{p}, 0))_- - (\mathbf{Y}_{\bar{p}}(p, 0))_- + [(p, 0)_-, (\bar{p}, 0)_-] \right) \Psi \\ &= - \left((\mathbf{Y}_p(0, \bar{p}))_- - (\mathbf{Y}_{\bar{p}}(0, p))_- - [(0, p)_-, (0, \bar{p})_-] \right) \Psi \\ &= \left(((\mathbf{Y}_p(\bar{p}))_\ell - (\mathbf{Y}_{\bar{p}}(p))_\ell + [p_\ell, \bar{p}_\ell]) \Psi_1, \right. \\ &\quad \left. - ((\mathbf{Y}_p(\bar{p}))_u - (\mathbf{Y}_{\bar{p}}(p))_u - [p_u, \bar{p}_u]) \Psi_2 \right) = \mathbf{Y}_{\{p, \bar{p}\}} \Psi \end{aligned}$$

where

$$(2.5) \quad \{p, \bar{p}\} := -\mathbf{Y}_p(\bar{p}) + \mathbf{Y}_{\bar{p}}(p) - [p_\ell, \bar{p}_\ell] + [p_u, \bar{p}_u].$$

We verify that these more general vector fields are indeed symmetries, i.e.

$$(i) \quad \left[\mathbf{Y}_p, \frac{\partial}{\partial t_n} \right] = 0, \quad (ii) \quad \left[\mathbf{Y}_p, \frac{\partial}{\partial s_n} \right] = 0,$$

using the argument of (1.23). For the case (i), set

$$Q = (q, 0) = (L_1^n, 0) \quad P = (p, 0),$$

and, using (2.1) and (2.2), we define

$$(a) \quad \tilde{\mathbf{Z}}_Q(P) = \frac{\partial}{\partial t_n}(p, 0) = [Q_+, P] \text{ and } \tilde{\mathbf{Z}}_Q \Psi = Q_+ \Psi$$

$$(b) \quad \tilde{\mathbf{Y}}_P(Q) = (-[p_\ell, q], 0) = [-(p_\ell, -p_u), (q, 0)] = [-P_-, Q], \text{ and } \tilde{\mathbf{Y}}_P \Psi = -P_- \Psi;$$

we are exactly in the situation (1.21) and (1.23), leading at once to $[\mathbf{Y}_P, \mathbf{Z}_Q] = 0$, which is (i). A similar argument leads to (ii)

Let \mathcal{A} be the Lie algebra of generalized symmetries of the 2-Toda vector fields, with the standard commutator $[\cdot, \cdot]$. Also, the ring of polynomials $\mathbf{C}[L_1, L_2, M_1, M_2]$ in the (noncommutative) variables L_1, L_2, M_1, M_2 , forms a Lie algebra, not only with regard to the standard bracket $[\cdot, \cdot]$, but also with regard to the new bracket $\{p, \bar{p}\}$ defined in (2.5), as will be seen in the next theorem; denote by

$A_1, B_1, C_1, \dots = \text{any monomial } (L_1, M_1, I) \text{ and } A_2, B_2, C_2, \dots = \text{any monomial } (L_2, M_2, I).$

Lemma 2.1. *The two brackets $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ of the Lie algebra $\mathbf{C}[L_1, L_2, M_1, M_2]$ interact as follows: for monomials $p = A_1 A_2 B_1 B_2 C_1 C_2 \dots$ and $\bar{p} = \bar{A}_1 \bar{A}_2 \bar{B}_1 \bar{B}_2 \bar{C}_1 \bar{C}_2 \dots$ in the algebra, two different expressions hold,*

$$(2.6) \quad \begin{aligned} \{p, \bar{p}\} &= [p, \bar{p}] - \bar{A}_1[p_1, \bar{A}_2]\bar{B}_1\bar{B}_2\bar{C}_1\bar{C}_2\dots - \bar{A}_1\bar{A}_2\bar{B}_1[p, \bar{B}_2]\bar{C}_1\bar{C}_2\dots - \dots \\ &\quad + A_1[\bar{p}, A_2]B_1B_2C_1C_2\dots + A_1A_2B_1[\bar{p}, B_2]C_1C_2\dots + \dots \\ &= [p, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2\bar{C}_1\bar{C}_2\dots + \bar{A}_1\bar{A}_2[p, \bar{B}_1]\bar{B}_2\bar{C}_1\bar{C}_2\dots + \dots \\ &\quad + A_1[\bar{p}, A_2]B_1B_2C_1C_2\dots + A_1A_2B_1[\bar{p}, B_2]C_1C_2\dots + \dots \end{aligned}$$

Theorem 2.2. *There is a Lie algebra homomorphism*

$$\mathbf{C}[L_1, L_2, M_1, M_2], \{ \cdot, \cdot \} \longmapsto \mathcal{A}, [\cdot, \cdot]$$

$$p \rightarrow \mathbf{Y}_p : \mathbf{Y}_p \Psi = -(p, 0)_- \Psi = (0, p)_- \Psi$$

i.e.,

$$[\mathbf{Y}_p, \mathbf{Y}_{\bar{p}}]\Psi = \mathbf{Y}_{\{p, \bar{p}\}}\Psi.$$

Moreover, we have the following inclusion (Lie algebra (anti)-homomorphism)

$$\mathcal{A} \supset w_\infty \oplus w_\infty \text{ with } \{A_1, A_2\} = 0,$$

and

$$\{A_1, B_1\} = [A_1, B_1] \quad \{A_2, B_2\} = -[A_2, B_2];$$

Also

$$(2.7) \quad \mathcal{A} \supset w_\infty \otimes \mathbf{C}(h, h^{-1}), \quad (\text{Loop algebra over } w_\infty),$$

in two different ways:

$$(2.8) \quad \{A_1 L_2^k, B_1 L_2^\ell\} = [A_1, B_1] L_2^{k+\ell}, \{L_1^k A_2, L_1^\ell B_2\} = -L_1^{k+\ell} [A_2, B_2].$$

Corollary 2.2.1. *The vector fields*

$$\mathbf{Y}_p \Psi = -(p, 0)_- \Psi \text{ with } p = M_1^n L_1^{n+k}, M_2^n L_2^{n+k}, L_1^\alpha L_2^\beta$$

all commute with the Toda vector fields $\partial/\partial t_n$ and $\partial/\partial s_n$. The vector fields $\partial/\partial c_{\alpha,\beta}$, corresponding to $p = L_1^\alpha L_2^\beta$, all commute among themselves.

Remark 2.2.2: The τ -function can then be considered as a function of t, s and c , as follows:

$$\tau(c_{10} + t_1, c_{20} + t_2, \dots; c_{01} + s_1, c_{02} + s_2, \dots; c_{11}, c_{12}, c_{21}, \dots).$$

Proof of Lemma 2.1, Theorem 2.2 and Corollary 2.2.1.: Since $\mathbf{Y}_p(\cdot)$ and $[x, \cdot]$ are derivations, one computes for $p = A_1 A_2 B_1 B_2$ and $\bar{p} = \bar{A}_1 \bar{A}_2 \bar{B}_1 \bar{B}_2$, that

$$\begin{aligned} \mathbf{Y}_p(\bar{p}) &= \mathbf{Y}_p(\bar{A}_1) \bar{A}_2 \bar{B}_1 \bar{B}_2 + \bar{A}_1 \mathbf{Y}_p(\bar{A}_2) \bar{B}_1 \bar{B}_2 + \bar{A}_1 \bar{A}_2 \mathbf{Y}_p(\bar{B}_1) \bar{B}_2 + \bar{A}_1 \bar{A}_2 \bar{B}_1 \mathbf{Y}_p(\bar{B}_2) \\ &= -[p_\ell, \bar{A}_1] \bar{A}_2 \bar{B}_1 \bar{B}_2 + \bar{A}_1 [p_u, \bar{A}_2] \bar{B}_1 \bar{B}_2 - \bar{A}_1 \bar{A}_2 [p_\ell, \bar{B}_1] \bar{B}_2 + \bar{A}_1 \bar{A}_2 \bar{B}_1 [p_u, \bar{B}_2] \\ \text{or } \left\{ \begin{aligned} &= -[p_\ell, \bar{A}_1 \bar{A}_2 \bar{B}_1 \bar{B}_2] + \bar{A}_1 [p_\ell + p_u, \bar{A}_2] \bar{B}_1 \bar{B}_2 + \bar{A}_1 \bar{A}_2 \bar{B}_1 [p_\ell + p_u, \bar{B}_2] \\ &= [p_u, \bar{A}_1 \bar{A}_2 \bar{B}_1 \bar{B}_2] - [p_\ell + p_u, \bar{A}_1] \bar{A}_2 \bar{B}_1 \bar{B}_2 - \bar{A}_1 \bar{A}_2 [p_\ell + p_u, \bar{B}_1] \bar{B}_2 \end{aligned} \right. \\ \text{or } \left\{ \begin{aligned} &= -[\bar{p}_\ell, \bar{p}] + \bar{A}_1 [p, \bar{A}_2] \bar{B}_1 \bar{B}_2 + \bar{A}_1 \bar{A}_2 \bar{B}_1 [p, \bar{B}_2] \\ &= [p_u, \bar{p}] - [p, \bar{A}_1] \bar{A}_2 \bar{B}_1 \bar{B}_2 - \bar{A}_1 \bar{A}_2 [p, \bar{B}_1] \bar{B}_2 \end{aligned} \right. \end{aligned}$$

and by symmetry, using the first expression, one finds

$$\mathbf{Y}_{\bar{p}}(p) = -[\bar{p}_\ell, p] + A_1 [\bar{p}, A_2] B_1 B_2 + A_1 A_2 B_1 [\bar{p}, B_2].$$

Using the definition (2.5) of $\{p, \bar{p}\}$, one computes

$$\begin{aligned}
\{p, \bar{p}\} &= -\mathbf{Y}_p(\bar{p}) + \mathbf{Y}_{\bar{p}}(p) - [p_\ell, \bar{p}_\ell] + [p_u, \bar{p}_u] \\
\text{or } \left\{ \begin{aligned} &= [p_\ell, \bar{p}] + [p, \bar{p}_\ell] - [p_\ell, \bar{p}_\ell] + [p_u, \bar{p}_u] \\ &\quad - \bar{A}_1[p, \bar{A}_2]\bar{B}_1\bar{B}_2 - \bar{A}_1\bar{A}_2\bar{B}_1[p, \bar{B}_2] + A_1[\bar{p}, A_2]B_1B_2 + A_1A_2B_1[\bar{p}, B_2] \end{aligned} \right. \\
&\quad \left\{ \begin{aligned} &= -[p_u, \bar{p}] + [p, \bar{p}_\ell] - [p_\ell, \bar{p}_\ell] + [p_u, \bar{p}_u] \\ &\quad + [p, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2[p, \bar{B}_1]\bar{B}_2 + A_1[\bar{p}, A_2]B_1B_2 + A_1A_2B_1[\bar{p}, B_2] \end{aligned} \right. \\
\text{or } \left\{ \begin{aligned} &= [p, \bar{p}] - \bar{A}_1[p, \bar{A}_2]\bar{B}_1\bar{B}_2 + A_1[\bar{p}, A_2]B_1B_2 - \bar{A}_1\bar{A}_2\bar{B}_1[p, \bar{B}_2] + A_1A_2B_1[\bar{p}, B_2] \\ &= [p, \bar{A}_1]\bar{A}_2\bar{B}_1\bar{B}_2 + \bar{A}_1\bar{A}_2[p, \bar{B}_1]\bar{B}_2 + A_1[\bar{p}, A_2]B_1B_2 + A_1A_2B_1[\bar{p}, B_2], \end{aligned} \right.
\end{aligned}$$

using

$$[p_\ell, \bar{p}] + [p, \bar{p}_\ell] - [p_\ell, \bar{p}_\ell] + [p_u, \bar{p}_u] = [p_\ell, \bar{p}] + [p_u, \bar{p}_\ell] + [p_u, \bar{p}_u] = [p_\ell, \bar{p}] + [p_u, \bar{p}] = [p, \bar{p}]$$

and

$$[p_u, \bar{p}] - [p, \bar{p}_\ell] + [p_\ell, \bar{p}_\ell] - [p_u, \bar{p}_u] = 0,$$

establishing (2.6).

In particular we have the following:

$$\begin{aligned}
(2.9) \quad \{A_1A_2, B_1B_2\} &= [A_1, B_1]A_2B_2 - A_1B_1[A_2, B_2] \\
\{A_2A_1, B_2B_1\} &= A_2B_2[A_1, B_1] - [A_2, B_2]A_1B_1 \\
\{A_1A_2, B_2B_1\} &= [A_1, B_2][B_1, A_2] - A_1[A_2, B_2]B_1 + B_2[A_1, B_1]A_2,
\end{aligned}$$

from which it follows at once that

$$(2.10) \quad \{A_1, A_2\} = 0, \quad \{A_1, B_1\} = [A_1, B_1], \quad \{A_2, B_2\} = -[A_2, B_2];$$

this implies that $w_\infty \oplus w_\infty \subset \mathcal{A}$. Moreover, the first relation of (2.9) leads to (2.8), implying (2.7) with $h = L_1$ or $h = L_2$, ending the proof of theorem 2.2. From (2.9), it follows that

$$\begin{aligned}
(2.11) \quad \{L_1^i L_2^j, L_1^{i'} L_2^{j'}\} &= \{L_2^j L_1^i, L_2^{j'} L_1^{i'}\} = 0 \\
\{L_1^i L_2^j, L_2^{i'} L_1^{j'}\} &= [L_1^i, L_2^{i'}][L_1^{j'}, L_2^j],
\end{aligned}$$

establishing corollary 2.2.1.

Theorem 2.3. *Given functions*

$$f(L_1, L_2) = \sum a_{ij} L_1^i L_2^j \text{ and } g(L_1, L_2) = \sum b_{ij} L_1^i L_2^j,$$

then the following matrices form a Virasoro algebra (without central extension) for the $\{ , \}$ -bracket¹⁰:

$$(2.12) \quad \{L_1^{i+1}(M_1 + f), L_1^{j+1}(M_1 + f)\} = (i - j)L_1^{i+j+1}(M_1 + f)$$

$$(2.13) \quad \{(M_2 - g)L_2^{i+1}, (M_2 - g)L_2^{j+1}\} = (j - i)(M_2 - g)L_2^{i+j+1}.$$

If $\frac{\partial g}{\partial z_1} = \frac{\partial f}{\partial z_2}$, then the two representations decouple

$$(2.14) \quad \{L_1^{i+1}(M_1 + f), (M_2 - g)L_2^{j+1}\} = 0.$$

Proof: At first, we need to prove

(2.15)

$$\begin{aligned} \{L_1^k M_1, L_1^\alpha f L_2^\beta\} &= -\alpha L_1^{k+\alpha-1} f L_2^\beta - L_1^{k+\alpha} \frac{\partial f}{\partial z_1} L_2^\beta \\ \{M_2 L_2^k, L_1^\alpha f L_2^\beta\} &= \beta L_1^\alpha f L_2^{k+\beta-1} + L_1^\alpha \frac{\partial f}{\partial z_2} L_2^{k+\beta}. \end{aligned}$$

Indeed, it suffices to do so for a monomial $f = L_1^i L_2^j$; e.g., using (2.9), we have

$$\begin{aligned} \{L_1^k M_1, L_1^{\alpha+i} L_2^{\beta+j}\} &= [L_1^k M_1, L_1^{\alpha+i}] L_2^{\beta+j} = -(\alpha + i) L_1^{\alpha+i+k-1} L_2^{\beta+j} \\ \{M_2 L_2^k, L_1^{\alpha+i} L_2^{\beta+j}\} &= -L_1^{\alpha+i} [M_2 L_2^k, L_2^{\beta+j}] = (\beta + j) L_1^{\alpha+i} L_2^{\beta+j+k-1}. \end{aligned}$$

Using the relation (2.10), (2.11) and (2.15) above, we find

$$\begin{aligned} &\{L_1^{i+1}(M_1 + f), L_1^{j+1}(M_1 + f)\} \\ &= \{L_1^{i+1} M_1, L_1^{j+1} M_1\} + \{L_1^{i+1} f, L_1^{j+1} M_1\} + \{L_1^{i+1} M_1, L_1^{j+1} f\} + \{L_1^{i+1} f, L_1^{j+1} f\} \\ &= (i - j) L_1^{i+j+1} M_1 + (i + 1) L_1^{i+j+1} f + L_1^{i+j+2} \frac{\partial f}{\partial z_1} - (j + 1) L_1^{i+j+1} f - L_1^{i+j+2} \frac{\partial f}{\partial z_1} \\ &= (i - j) L_1^{i+j+1} (M_1 + f), \end{aligned}$$

and the same relations yield

¹⁰ $f = f(L_1, L_2) = \sum_{i,j} a_{ij} L_1^i L_2^j$ and $g = g(L_1, L_2) = \sum_{i,j} b_{ij} L_1^i L_2^j$

$$\begin{aligned}
& \{(M_2 - g)L_2^{i+1}, (M_2 - g)L_2^{j+1}\} \\
&= \{M_2L_2^{i+1}, M_2L_2^{j+1}\} - \{gL_2^{i+1}, M_2L_2^{j+1}\} - \{M_2L_2^{i+1}, gL_2^{j+1}\} + \{gL_2^{i+1}, gL_2^{j+1}\} \\
&= (j - i)M_2L_2^{i+j+1} + (i + 1)gL_2^{i+j+1} + \frac{\partial g}{\partial z_2}L_2^{i+j+2} - (j + 1)gL_2^{i+j+1} - \frac{\partial g}{\partial z_2}L_2^{i+j+2} \\
&= (j - i)(M_2 - g)L_2^{i+j+1},
\end{aligned}$$

confirming (2.12) and (2.13). Finally, one checks (2.14),

$$\begin{aligned}
& \{L_1^{i+1}(M_1 + f), (M_2 - g)L_2^{j+1}\} \\
&= \{L_1^{i+1}M_1, M_2L_2^{j+1}\} + \{L_1^{i+1}f, M_2L_2^{j+1}\} - \{L_1^{i+1}M_1, gL_2^{j+1}\} - \{L_1^{i+1}f, gL_2^{j+1}\} \\
&= 0 - L_1^{i+1}\frac{\partial f}{\partial z_2}L_2^{j+1} + L_1^{i+1}\frac{\partial g}{\partial z_1}L_2^{j+1} + 0 = -L_1^{i+1}\left(\frac{\partial f}{\partial z_2} - \frac{\partial g}{\partial z_1}\right)L_2^{j+1},
\end{aligned}$$

ending the proof of Theorem 2.3.

In the next theorem, we provide the explicit solution to a large class of symmetries and, in particular, to the system of differential equations (0.3'). Since the S_1 and S_2 of the Borel decomposition of the moment matrix m_∞ will turn out to be the S_1 and S_2 of this section, it will suffice to prove theorem 2.4. Note that symmetry vector fields of the form

$$(2.16) \quad \frac{\partial \Psi}{\partial c} := \mathbf{Y}_p \Psi = -(p, 0)_- \Psi = (-p_\ell, p_u)(\Psi_1, \Psi_2) \text{ with } p = p_1(L_1, M_1)p_2(L_2, M_2)$$

induce vector fields on $S_1^{-1}S_2$:

$$(2.17) \quad \frac{\partial S_1^{-1}S_2}{\partial c} = p_1(\Lambda, \varepsilon + \sum_{k=1}^{\infty} kt_k \Lambda^{k-1})S_1^{-1}S_2p_2(\Lambda^\top, \varepsilon^* + \sum_{k=1}^{\infty} ks_k \Lambda^{\top k-1}).$$

Theorem 2.4. *For symmetries of the form (2.16), we find the solution*

$$(2.18) \quad (S_1^{-1}S_2)(t, s, c) = \sum_{r=0}^{\infty} \frac{c^r}{r!} p_1^r\left(\Lambda, \varepsilon + \sum_1^{\infty} kt_k \Lambda^{k-1}\right) e^{\sum_1^{\infty} t_k \Lambda^k} (S_1^{-1}S_2)(0, 0, 0) e^{-\sum_1^{\infty} s_k \Lambda^{\top k}} p_2^r(\Lambda^\top, \varepsilon^* + \sum_1^{\infty} kt_k \Lambda^{\top k-1})$$

with

$$(2.19) \quad \tau_N(t, s, c) = \det\left(S_1^{-1}S_2(t, s, c)_{ij}\right)_{0 \leq i, j \leq N-1}.$$

Corollary 2.4.1. *In particular, symmetries of the form*

$$\frac{\partial \Psi}{\partial c_{\alpha\beta}} := \mathbf{Y}_{L_1^\alpha L_2^\beta} \Psi,$$

have the following solution

$$S_1^{-1} S_2(t, s, c_{\alpha\beta}) = \sum_{r=0}^{\infty} \frac{c_{\alpha\beta}^r}{r!} \Lambda^{r\alpha} e^{\sum_1^\infty t_k \Lambda^k} S_1^{-1} S_2(0, 0, 0) e^{-\sum_1^\infty s_k \Lambda^{\top k}} (\Lambda^\top)^{r\beta}$$

Corollary 2.4.2. *In general, setting $t = c_{10}$ and $s = -c_{0,1}$, we have*

$$S_1^{-1} S_2(t, s, c_{11}, c_{12}, \dots) = \sum_{\substack{(r_{\alpha\beta})_{\alpha, \beta \geq 0} \in \mathbf{Z}^\infty \\ (\alpha, \beta) \neq (0, 0)}} \left(\prod_{(\alpha, \beta)} \frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!} \right) \Lambda^{\sum_{\alpha \geq 1} \alpha r_{\alpha\beta}} S_1^{-1} S_2(0, 0, 0, \dots) \Lambda^{\top \sum_{\beta \geq 1} \beta r_{\alpha\beta}};$$

Proof of theorem 2.4: Note that the vector field (2.16) induces the vector field

$$\mathbf{Y}_p(S_1) = -p_\ell S_1 \text{ and } \mathbf{Y}_p(S_2) = p_u S_2$$

and thus for $p = p_1(L_1, M_1)p_2(L_2, M_2)$ as in (2.16), we find

$$\begin{aligned} \mathbf{Y}_p(S_1^{-1} S_2) &= -S_1^{-1}(-p_\ell S_1) S_1^{-1} S_2 + S_1^{-1} p_u S_2 = S_1^{-1} p S_2 \\ &= S_1^{-1} p S_2 \\ &= p_1(\Lambda, \varepsilon + \sum_1^\infty k t_k \Lambda^{k-1}) S_1^{-1} S_2 p_2(\Lambda^\top, \varepsilon^* + \sum_1^\infty k s_k (\Lambda^\top)^{k-1}), \end{aligned}$$

a linear differential equation for $S_1^{-1} S_2$. Since for any linear differential equation

$$\frac{\partial x}{\partial c} = L(x) \iff x(c) = e^{cL} x(0) = \sum \frac{c^r}{r!} L^r x(0),$$

the conclusion (2.18) follows. Corollary 2.4.1. is merely a special case of (2.18), while corollary 2.4.2 takes into account the t -, s - and $c_{\alpha\beta}$ -flows all at once.

3 Borel decomposition of moment matrices, τ -functions and string-orthogonal polynomials

Consider the weight $\rho(y, z)dydz = e^{V(y, z)}dydz$ on \mathbb{R}^2 , the corresponding inner product $\langle \cdot, \cdot \rangle$ and the moment matrix¹¹ $m_n(t, s, c)$, as in (0.1), (0.2) and (0.3). In the introduction, *string-orthogonal polynomials* were defined as two sequences of monic polynomials of degree i , each depending on one variable (y and $z \in \mathbb{R}$)

$$\{p_i^{(1)}(y)\}_{i=0}^{\infty} \quad \text{and} \quad \{p_i^{(2)}(z)\}_{i=0}^{\infty},$$

orthogonal in the following sense

$$\langle p_i^{(1)}, p_j^{(2)} \rangle = h_i \delta_{ij}.$$

Also consider the $N \times \infty$ matrix of Schur polynomials¹² $p_i(t)$:

$$\begin{aligned} E_N(t) &= \left(\begin{array}{cccc|cc} 1 & p_1(t) & p_2(t) & \dots & p_{N-1}(t) & p_N(t) & \dots \\ 0 & 1 & p_1(t) & \dots & p_{N-2}(t) & p_{N-1}(t) & \dots \\ \vdots & \vdots & & & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & p_1(t) & p_2(t) & \dots \\ 0 & 0 & 0 & \dots & 1 & p_1(t) & \dots \end{array} \right) \\ &= (p_{j-i}(t))_{\substack{0 \leq i \leq N-1 \\ 0 \leq j < \infty}} \end{aligned}$$

The polynomials $p_n^{(1)}(y)$ and $p_n^{(2)}(z)$ have explicit expressions in terms of the moments

$$(3.1) \quad p_n^{(1)}(y) = \frac{1}{\det m_n(t, s, c)} \det \left(\begin{array}{ccc|c} & & & 1 \\ & & & y \\ & & & \vdots \\ \hline \mu_{n,0}(t, s, c) & \dots & \mu_{n,n-1}(t, s, c) & y^n \end{array} \right)$$

¹¹In this and subsequent sections

$$(t, s, c) = \left((t_n)_{n \geq 1}, (s_n)_{n \geq 1}, (c_{\alpha\beta})_{\alpha, \beta \geq 1} \right) =: (c_{\alpha\beta})_{\substack{\alpha, \beta \geq 0 \\ (\alpha, \beta) \neq (0, 0)}}$$

with the understanding that $t_n = c_{n0}$, $s_n = -c_{0n}$.

¹²The Schur polynomials p_i , defined by $e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t) z^k$ and $p_k(t) = 0$ for $k < 0$, are not to be confused with the string-orthogonal polynomials $p_i^{(k)}$, $k = 1, 2$.

and

$$p_n^{(2)}(z) = \frac{1}{\det m_n(t, s, c)} \det \left(\begin{array}{ccc|c} & & & 1 \\ & & & z \\ & & & \vdots \\ & & & z^n \\ \hline \mu_{0,n}(t, s, c) & \dots & \mu_{n-1,n}(t, s, c) & \end{array} \right),$$

from which

$$h_i = \langle p_i^{(1)}, p_i^{(2)} \rangle = \langle p_i^{(1)}, z^i \rangle = \frac{\det m_{i+1}}{\det m_i}.$$

For future use, we need some notation: for an arbitrary matrix

$$A = (a_{ij})_{0 \leq i, j \leq N-1}$$

with N finite or infinite, denote by A_i the upper-left $i \times i$ minor and $\Delta_i := \det A_i$, with $i \geq 1$ and $\Delta_0 = 1$; assuming all $\Delta_i \neq 0$, define the following monic polynomials of degree i ($0 \leq i \leq N-1$)

$$(3.2) \quad P_i(z, A) := \frac{1}{\Delta_i} \det \begin{pmatrix} & & 1 \\ & A_i & \vdots \\ & & z^{i-1} \\ a_{i0} & \dots & a_{i,i-1} & z^i \end{pmatrix} =: \sum_{0 \leq j \leq i} \frac{\Delta_{ji}^{(i+1)}}{\Delta_i} z^j = z^i + \dots,$$

with

$$\Delta_{ii}^{(i+1)} = \Delta_i \text{ and } \Delta_{ji}^{(i+1)} = 0, \quad i < j.$$

Given the character $\bar{\chi}(z) = (1, z, z^2, \dots)^\top$, the lower triangular matrix $S(A)$, with 1's along the diagonal, is defined such that

$$(3.3) \quad S(A)\chi(z) := (P_0(z, A), P_1(z, A), \dots)^\top.$$

Also define the $N \times N$ diagonal matrix $h = h(A)$

$$(3.4) \quad h := \text{diag}(h_0, h_1, \dots, h_{N-1}) := \text{diag}\left(\frac{\Delta_1}{\Delta_0}, \dots, \frac{\Delta_N}{\Delta_{N-1}}\right),$$

and so $\Delta_N = \prod_0^{N-1} h_i$. Theorem 3.1 shows that performing the Borel decomposition of the matrix m_∞ is tantamount to the construction of the string-orthogonal polynomials. Given m_∞ , define $S(m_\infty)$, $S(m_\infty^\top)$ and $h(m_\infty)$ by means of (3.3) and (3.4). We now state Theorem 3.1, whose proof can be found in [AvM3].

Theorem 3.1. *The vectors of string-orthogonal polynomials are given by*

$$(3.5) \quad p^{(1)}(y) = L^{-1}\chi(y) \text{ and } p^{(2)}(z) = (U^{-1})^\top \chi(z)$$

where L and U are the lower- and upper-triangular matrices respectively, with 1's on the diagonal, appearing in the Borel decomposition of the matrix m_∞ :

$$(3.6) \quad m_\infty = L h U := S^{-1}(m_\infty)h(m_\infty)(S^{-1}(m_\infty^\top))^\top.$$

Corollary 3.1.1. *The string-orthogonality relations are equivalent to the Borel decomposition of the moment matrix m_∞ :*

$$(3.7) \quad \langle p_i^{(1)}, p_j^{(2)} \rangle = h_i \delta_{ij} \iff S(m_\infty)m_\infty(S(m_\infty^\top))^\top = h(m_\infty).$$

Remark 3.1.1: Any matrix Δ (finite or infinite), with all $\Delta_i \neq 0$, can be realized as a moment matrix m_∞ and the recipe (3.6) provides its Borel decomposition.

The determinant of the moment matrix m_N can be expressed in terms of a double matrix integral, as stated in the following theorem:

Theorem 3.2. *The following integral*

$$(3.8) \quad \begin{aligned} \tau_N(t, s, c) &:= \int \int_{\vec{u}, \vec{v} \in \mathbb{R}^N} e^{\sum_{k=1}^N V(u_k, v_k)} \prod_{i < j} (u_i - u_j) \prod (v_i - v_j) d\vec{u} d\vec{v} \\ &= N! \det m_N(t, s, c) \\ &= N! \det \left(E_N(t) m_\infty(0, 0, c) E_N(-s)^\top \right) \end{aligned}$$

is a τ -function of t and $-s$; in particular τ_N satisfies the KP-hierarchy in t and s , for each $N = 0, 1, \dots$; moreover the $h_n = \langle p_n^{(1)}, p_n^{(2)} \rangle$ are given by

$$(3.9) \quad h_n(t, s, c) = \frac{\tau_{n+1}(t, s, c)}{\tau_n(t, s, c)} \quad \tau_0(t, s, c) = 1.$$

Remark: For $V(y, z) = \sum_1^\infty t_i y^i + c y z - \sum_1^\infty s_i z^i$, the integral (3.8) is well known [AS1] to be the two matrix integral, integrated over the space of Hermitean matrices of size N :

$$\left(\frac{\pi}{\sqrt{c}} \right)^{-N^2+N} \cdot \prod_{k=1}^N k! \int \int_{\mathcal{H}_N \times \mathcal{H}_N} dM_1 dM_2 e^{\text{tr } V(M_1, M_2)}.$$

Before giving the proof of this theorem, we shall need several Lemmas:

Lemma 3.2.1.

$$m_N(t, s) = E_N(t) m_\infty(0, 0) E_N(-s)^\top.$$

and

$$m_\infty(t, s) = e^{\sum_1^\infty t_n \Lambda^n} m_\infty(0, 0) e^{-\sum_1^\infty s_n \Lambda^\top n}.$$

Proof:

$$\begin{aligned} m_N(t, s) &= \left(\int \int_{\mathbb{R}^2} u^{\ell-1} v^{k-1} e^{\sum t_i u^i} e^{V_{12}(u, v)} e^{-\sum s_j v^j} du dv \right)_{1 \leq \ell, k \leq N} \\ &= \left(\sum_{i, j=0}^\infty p_i(t) \left(\int \int_{\mathbb{R}^2} e^{V_{12}(u, v)} u^{i+\ell-1} v^{j+k-1} du dv \right) p_j(-s) \right)_{1 \leq \ell, k \leq N} \\ &= \left(\sum_{i, j=0}^\infty p_i(t) \mu_{i+\ell-1, j+k-1}(0, 0) p_j(-s) \right)_{1 \leq \ell, k \leq N} \\ &= E_N(t) m_\infty(0, 0) E_N(-s)^\top. \end{aligned}$$

The second line follows from the first one, since $E_\infty(t) = e^{\sum t_n \Lambda^n}$, establishing Lemma 3.2.1.

Lemma 3.2.2. *For arbitrary sequences of monic polynomials $p_k^{(1)}$ and $p_k^{(2)}$, we have*

$$\begin{aligned} &\det(u_k^{\ell-1})_{1 \leq \ell, k \leq N} \det(v_k^{\ell-1})_{1 \leq \ell, k \leq N} \\ &= \sum_{\sigma \in S_N} \det(u_{\sigma(k)}^{\ell-1} v_{\sigma(k)}^{k-1})_{1 \leq \ell, k \leq N} \\ &= \det(p_{\ell-1}^{(1)}(u_k))_{1 \leq \ell, k \leq N} \det(p_{\ell-1}^{(2)}(v_k))_{1 \leq \ell, k \leq N}. \end{aligned}$$

Proof: This is a standard result about Vandermonde determinants.

Consider a linear space W^0 spanned by formal (Laurent) series in z (for large z):

$$W^0 = \text{span} \left\{ \sum_{j=-\infty}^{N-1} a_{jk} z^j, k = 0, \dots, N-1 \right\} \oplus z^N H_+.$$

Lemma 3.2.3. *Letting*

$$W^t = e^{\sum t_i z^i} W^0,$$

the τ -function

$$\tau(t) = \det(\text{Proj} : W^t \rightarrow H_+)$$

has the form

$$\tau(t) = \det(E_N(t)a_N),$$

where a_N is the (∞, N) -matrix of coefficients

$$(3.10) \quad a_N = \begin{pmatrix} a_{N-1,0} & \cdots & a_{N-1,N-1} \\ \vdots & & \vdots \\ a_{10} & \cdots & a_{1,N-1} \\ a_{00} & \cdots & a_{0,N-1} \\ \hline a_{-10} & \cdots & a_{-1,N-1} \\ a_{-20} & \cdots & a_{-2,N-1} \\ \vdots & & \vdots \end{pmatrix}$$

Proof:

$$\begin{aligned} W^t &= \text{span} \left\{ \sum_{i=0}^{\infty} p_i(t) z^i \sum_{j=-\infty}^{N-1} a_{jn} z^j, n = 0, 1, \dots, N-1 \right\} \oplus \left\{ \sum_0^{\infty} p_i(t) z^{i+N+k}, k = 0, 1, 2, \dots \right\} \\ &= \text{span} \left\{ \sum_{k \in \mathbf{Z}} z^k \sum_{j \geq \max(-k, -N+1)} p_{k+j}(t) a_{-jn}, n = 0, 1, \dots, N-1 \right\} \oplus \\ &\quad \left\{ \sum_0^{\infty} p_i(t) z^{i+N+k}, k = 0, 1, 2, \dots \right\}. \end{aligned}$$

Therefore, recording the coefficients of z^k in the k th row starting from the bottom ($k = 0, 1, 2, \dots$), the projection $W^t \rightarrow H_+$ has the following matrix representation:

$$\begin{aligned} \tau(t) &= \det \text{Proj}(W^t \rightarrow H_+) \\ &= \det \left(\begin{array}{ccc|ccc} & & & \vdots & \vdots & \vdots \\ & & & p_2 & p_1 & 1 & \cdots \\ & & \vdots & p_1 & 1 & 0 & \cdots \\ \sum_{j \geq -N+1} a_{-j0} p_{j+N} & \cdots & \sum_{j \geq -N+1} a_{-j,N-1} p_{j+N} & 1 & 0 & 0 & \cdots \\ \hline \sum_{j \geq -N+1} a_{-j0} p_{j+N-1} & \cdots & \sum_{j \geq -N+1} a_{-j,N-1} p_{j+N-1} & & & & \\ \vdots & & \vdots & & & & \\ \sum_{j \geq -1} a_{-j0} p_{j+1} & \cdots & \sum_{j \geq -1} a_{-j,N-1} p_{j+1} & & & & \\ \sum_{j \geq 0} a_{-j0} p_j & \cdots & \sum_{j \geq 0} a_{-j,N-1} p_j & & & & \end{array} \right) \end{aligned}$$

$$\begin{aligned}
&= \det \begin{pmatrix} \sum_{j \geq -N+1} a_{-j0} p_{j+N-1} & \cdots & \sum a_{-j,N+1} p_{j+N-1} \\ \vdots & & \vdots \\ \sum_{j \geq 0} a_{-j0} p_j & & \sum_{j \geq 0} a_{-j,N-1} p_j \end{pmatrix} \\
&= \det(E_N a_N),
\end{aligned}$$

ending the proof of Lemma 3.3.

Proof of Theorem 3.2: Indeed, using Lemma 3.2.2 and Lemma 3.2.1, we have

(3.11)

$$\begin{aligned}
&\int \int_{u,v \in \mathbb{R}^N} e^{\sum_{k=1}^N V(u_k, v_k)} \prod_{i < j} (u_i - u_j) \prod_{i < j} (v_i - v_j) du_1 \dots du_N dv_1 \dots dv_N \\
&= \int \int_{u,v \in \mathbb{R}^N} \prod_{k=1}^N e^{V(u_k, v_k)} \sum_{\sigma \in S_N} \det(u_{\sigma(k)}^{\ell-1} v_{\sigma(k)}^{k-1})_{1 \leq \ell, k \leq N} du_1 \dots du_N dv_1 \dots dv_N \\
&= \sum_{\sigma \in S_N} \det \left(\int \int_{\mathbb{R}^2} e^{V(u,v)} u^{\ell-1} v^{k-1} du dv \right)_{1 \leq \ell, k \leq N} \\
&= N! \det(m_N(t, s)) \\
&= N! \det(E_N(t) m_\infty(0, 0) E_N(-s)^\top).
\end{aligned}$$

Lemma 3.2.3 shows the latter expression is a τ -function in t and in $-s$, where alternately $a_N^{(1)}(-s) = m_\infty(0, 0) E_N^\top(-s)$ yields an initial plane in the Grassmannian parametrized by s , or $a_N^{(2)}(t) = m_\infty^\top(0, 0) E_N^\top(t)$ yields an initial plane parametrized by t . Finally (3.9) follows at once from (3.1'), ending the proof of Theorem 3.2.

4 From string-orthogonal polynomials to the two-Toda lattice and the string equation

Given the (t, s, c) -dependent weight $e^{V(y,z)}$ as in (0.1), its moment matrix m_∞ admits, according to Theorem 3.1, the Borel decomposition

$$(4.1) \quad m_\infty = S_1^{-1} h(\bar{S}_2^{-1})^\top,$$

upon setting (in the notation (3.3)),

$$(4.2) \quad S_1 := S(m_\infty), \quad \bar{S}_2 := S(m_\infty^\top).$$

Also, according to Theorem 3.1, the semi-infinite matrices $S_1 = S_1(t, s, c)$ and $\bar{S}_2 = \bar{S}_2(t, s, c) \in \mathcal{D}_{-\infty, 0}$ lead to string-orthogonal (monic) polynomials $(p^{(1)}, p^{(2)})$ with $\langle p_n^{(1)}, p_m^{(2)} \rangle = \delta_{n,m} h_n$; we also define new matrices $\bar{S}_1, S_2 \in \mathcal{D}_{0, \infty}$ and vectors Ψ_1 and Ψ_2^* , as follows:

$$\begin{aligned}
(4.3) \quad & p^{(1)}(z) =: S_1 \bar{\chi}(z) \quad \text{and} \quad p^{(2)}(z) =: \bar{S}_2 \bar{\chi}(z) \\
& \bar{S}_1 := h(S_1^{-1})^\top \quad \text{and} \quad S_2 := h(\bar{S}_2^{-1})^\top \\
& \Psi_1 := e^{\Sigma t_k z^k} p^{(1)}(z) \quad \text{and} \quad \Psi_2^* := e^{-\Sigma s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) \\
& \quad = e^{\Sigma t_k z^k} S_1 \bar{\chi}(z) \quad \quad \quad = e^{-\Sigma s_k z^{-k}} (S_2^{-1})^\top \bar{\chi}(z^{-1}).
\end{aligned}$$

Also define matrices $L_1, \bar{L}_2 \in \mathcal{D}_{-\infty, 1}$, $L_2, \bar{L}_1 \in \mathcal{D}_{-1, \infty}$, and $Q_1, \bar{Q}_2 \in \mathcal{D}_{-\infty, -1}$ so that

$$\begin{aligned}
(4.4) \quad & \text{(i)} \quad z p_n^{(1)}(z) = \sum_{\ell \leq n+1} (L_1)_{n\ell} p_\ell^{(1)}(z) \quad z p_n^{(2)}(z) = \sum_{\ell \leq n+1} (\bar{L}_2)_{n\ell} p_\ell^{(2)}(z) \\
& \text{(ii)} \quad \frac{\partial}{\partial z} p_n^{(1)}(z) = \sum_{\ell \leq n-1} (Q_1)_{n\ell} p_\ell^{(1)}(z) \quad \frac{\partial}{\partial z} p_n^{(2)}(z) = \sum_{\ell \leq n-1} (\bar{Q}_2)_{n\ell} p_\ell^{(2)}(z) \\
& \text{(iii)} \quad \bar{L}_1 = h L_1^\top h^{-1}, \quad \bar{L}_2 = h L_2^\top h^{-1}.
\end{aligned}$$

Using the latter, we check,

$$z \Psi_1 = e^{\Sigma t_k z^k} z p^{(1)}(z) = e^{\Sigma t_k z^k} L_1 p^{(1)} = L_1 \Psi_1,$$

and

$$z^{-1} \Psi_2^*(z) = e^{-\Sigma s_k z^{-k}} h^{-1} z^{-1} p^{(2)}(z^{-1}) = e^{-\Sigma s_k z^{-k}} h^{-1} \bar{L}_2 p^{(2)}(z^{-1}) = L_2^\top \Psi_2^*,$$

leading to the formula (4.5) below upon setting $L_2^* = L_2^\top$; at the same time, we also define the matrices M_1 and M_2^* :

$$\begin{aligned}
(4.5) \quad & (L_1, L_2^*)(\Psi_1, \Psi_2^*) = (z, z^{-1})(\Psi_1, \Psi_2^*) \\
& (M_1, M_2^*)(\Psi_1, \Psi_2^*) := \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z^{-1}} \right) (\Psi_1, \Psi_2^*).
\end{aligned}$$

Theorem 4.1 (“String equations”). *The matrices L and Q satisfy the constraints¹³,*

$$(4.6) \quad \text{(i)} \quad Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) = 0, \quad \text{(ii)} \quad \bar{Q}_2 + \left(\frac{\partial V}{\partial z}(\bar{L}_1^\top, \bar{L}_2^\top) \right)^\top = 0,$$

¹³If $f(y, z) = \Sigma c_{ij} y^i z^j$, then $f(L_1, L_2) \equiv \Sigma c_{ij} L_1^i L_2^j$.

and the matrices L and M ,

$$(4.7) \quad (i) \quad M_1 + \frac{\partial V_{12}}{\partial y}(L_1, L_2) = 0, \quad (ii) \quad M_2^* + \left(\frac{\partial V_{12}}{\partial z}(L_1, L_2) \right)^\top = 0.$$

Corollary 4.1.1. *If*

$$V(y, z) = \sum_1^{\ell_1} t_i y^i + cyz - \sum_1^{\ell_2} s_i z^i,$$

then L_1 is a $\ell_2 + 1$ -band matrix, having thus $\ell_2 - 1$ subdiagonals below the main diagonal; also, L_2 is a $\ell_1 + 1$ -band matrix, with $\ell_1 - 1$ subdiagonals above the main diagonal.

Theorem 4.2 (“Toda equations”). *The Borel decomposition of the moment matrix for the weight e^V*

$$(4.8) \quad m_\infty = S_1^{-1} h(\bar{S}_2^{-1})^\top = S_1^{-1} S_2$$

provides matrices S_i such that $L = (L_1, L_2) = (S_1 \Lambda S_1^{-1}, S_2 \Lambda^\top S_2^{-1})$. With regard to the parameters (t, s, c) appearing in the weight e^V , the matrices m_∞ and L satisfy the two-Toda equations and symmetry equations¹⁴

$$(4.9) \quad \frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty \Lambda^{\top n}, \quad \frac{\partial m_\infty}{\partial c_{\alpha\beta}} = \Lambda^\alpha m_\infty \Lambda^{\top \beta},$$

$$\frac{\partial L}{\partial t_n} = [(L_1^n, 0)_+, L] \quad \frac{\partial L}{\partial s_n} = [(0, L_2^n)_+, L], \quad \frac{\partial L}{\partial c_{\alpha,\beta}} = -[(L_1^\alpha L_2^\beta, 0)_-, L].$$

Corollary 4.2.1. *The moment matrix $m_\infty(t, s, c)$ can be expressed in terms of its initial value $m_\infty(0, 0, 0)$ as follows:*

$$m_\infty(t, s, c) = \sum_{\substack{(r_{\alpha\beta})_{\alpha,\beta \geq 0} \in \mathbf{Z}^\infty \\ (\alpha,\beta) \neq (0,0)}} \left(\prod_{(\alpha,\beta)} \frac{c_{\alpha\beta}^{r_{\alpha\beta}}}{r_{\alpha\beta}!} \right) \Lambda^{\sum_{\alpha \geq 1} \alpha r_{\alpha\beta}} m_\infty(0, 0, 0) \Lambda^\top \sum_{\beta \geq 1} \beta r_{\alpha\beta}.$$

¹⁴Note $\frac{\partial L}{\partial c_{\alpha\beta}} = -[(L_1^\alpha L_2^\beta, 0)_-, L] = [(0, L_1^\alpha L_2^\beta)_-, L]$.

Corollary 4.2.2. *The string-orthogonal polynomials, defined in (3.1) or (4.3), have the following expression in terms of the τ -function $\tau_n = \det m_n$ (see (0.3) and footnote 2):*

$$\begin{aligned} p_n^{(1)}(y) &= y^n \frac{\tau_n(t - [y^{-1}], s, c)}{\tau_n(t, s, c)} & p_n^{(2)}(z) &= z^n \frac{\tau_n(t, s + [z^{-1}], c)}{\tau_n(t, s, c)}. \\ &= \sum_{0 \leq k \leq n} \frac{p_{n-k}(-\tilde{\partial}_t) \det m_n(t, s, c)}{\det m_n(t, s, c)} y^k & &= \sum_{0 \leq k \leq n} \frac{p_{n-k}(\tilde{\partial}_s) \det m_n(t, s, c)}{\det m_n(t, s, c)} z^k. \end{aligned}$$

Theorem 4.3 (“Trace formula”). ([ASV4]) *The following holds for $n, m \geq -1$:*

$$\sum_{0 \leq i \leq N-1} (L_1^{n+1} L_2^{m+1})_{ii} = \frac{1}{\tau_N(t, s)} p_{n+N}(\tilde{\partial}_t) p_{m+N}(-\tilde{\partial}_s) \tau_1 \circ \tau_{N-1}.$$

Remark 1: Alternatively, also according to [ASV4], the above formula can be written in terms of derivatives of τ_{N-1} with coefficients involving the moments $\mu_{ij} = \langle y^i, z^j \rangle$, defined in (0.3):

$$\sum_{0 \leq i \leq N-1} (L_1^{n+1} L_2^{m+1})_{ii} = \frac{1}{\tau_N(t, s)} \sum_{\substack{i+i'=n+N \\ j+j'=m+N \\ i, i', j, j' \geq 0}} \mu_{ij} p_{i'}(-\tilde{\partial}_t) p_{j'}(\tilde{\partial}_s) \tau_{N-1}(t, s).$$

The reader will find the proof of Theorem 4.3 in [ASV4]. Before proving Theorems 4.1 and 4.2, we first need a few Lemmas:

Lemma 4.4. *The matrices $L_i, \bar{L}_i, M_1, M_2^*, Q_1, \bar{Q}_2$ satisfy:*

$$(4.10) \quad \begin{cases} L_1 = S_1 \Lambda S_1^{-1}, & \bar{L}_2 = \bar{S}_2 \Lambda \bar{S}_2^{-1} \\ \bar{L}_1 = \bar{S}_1 \Lambda^\top \bar{S}_1^{-1}, & L_2 = S_2 \Lambda^\top S_2^{-1} \end{cases}$$

and

$$(4.11) \quad M_1 = Q_1 + \frac{\partial V_1}{\partial y}(L_1), \quad M_2^* = h^{-1} \bar{Q}_2 h + \frac{\partial V_2}{\partial z}(L_2^\top).$$

Proof: From (4.3) and (4.4),

$$L_1 S_1 \bar{\chi}(z) = L_1 p_1 = z p_1 = z S_1 \bar{\chi}(z) = S_1 z \bar{\chi}(z) = S_1 \Lambda \bar{\chi}(z), \quad \text{for all } z,$$

together with $\bar{L}_1 = hL_1^\top h^{-1}$, implies the formulas (4.10) for L_1 and \bar{L}_1 ; a similar argument implies those for L_2 and \bar{L}_2 .

To prove (4.11), observe, from (4.3) and (4.4), that for $\Psi_1 = \Psi_1(t, s, c; z)$

$$\begin{aligned} M_1 \Psi_1 &= \frac{\partial \Psi_1}{\partial z} = \frac{\partial}{\partial z} \left(e^{\Sigma t_k z^k} p^{(1)}(z) \right) \\ &= \sum_{k \geq 1} k t_k z^{k-1} \Psi_1 + e^{\Sigma t_k z^k} \frac{\partial}{\partial z} p^{(1)}(z) \\ &= (\Sigma k t_k L_1^{k-1} + Q_1) \Psi_1, \\ &= \left(\frac{\partial V_1}{\partial y}(L_1) + Q_1 \right) \Psi_1, \end{aligned}$$

and similarly that for $\Psi_2^* = \Psi_2^*(t, s, c; z)$,

$$\begin{aligned} M_2^* \Psi_2^* &= \frac{\partial \Psi_2^*}{\partial z^{-1}} = \frac{\partial}{\partial z^{-1}} \left(e^{-\Sigma s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) \right) \\ &= -\Sigma k s_k z^{-k+1} \Psi_2^* + e^{-\Sigma s_k z^{-k}} h^{-1} \frac{\partial}{\partial z^{-1}} p^{(2)}(z^{-1}) \\ &= -\Sigma k s_k (L_2^\top)^{k-1} \Psi_2^* + e^{-\Sigma s_k z^{-k}} h^{-1} \bar{Q}_2 p^{(2)}(z^{-1}) \\ &= \left(\frac{\partial V_2(L_2^\top)}{\partial z} + h^{-1} \bar{Q}_2 h \right) \Psi_2^*, \end{aligned}$$

concluding the proof of Lemma 4.4.

Proof of Theorem 4.1: The following calculation, must be done in the following spirit: for arbitrarily large m , set all $t_i = 0$, for $i > 2m$ and $t_{2m} \neq 0$. Using in the fifth equality the fact that $c_{i0} = t_i$ and $c_{0i} = -s_i$ (see (4.1)), compute, for all integer n and $m \geq 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \frac{\partial}{\partial y} \left\{ e^{V_1(y)} p_n^{(1)}(y) \left(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V(y,z)-V_1(y)} dz \right) \right\} dy \\ &= \int_{\mathbb{R}} e^{V_1(y)} \left(\frac{\partial V_1}{\partial y}(y) p_n^{(1)}(y) + \frac{\partial p_n^{(1)}}{\partial y}(y) \right) \left(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V-V_1} dz \right) dy \\ &\quad + \int_{\mathbb{R}} e^{V_1(y)} p_n^{(1)}(y) \left(\int_{\mathbb{R}} p_m^{(2)}(z) e^{V-V_1} \frac{\partial}{\partial y} (V - V_1) dz \right) dy \\ &= \int_{\mathbb{R}} \sum_{\ell} (Q_1)_{n\ell} p_\ell^{(1)}(y) \left(\int_{\mathbb{R}} p_m^{(2)}(z) e^V dz \right) dy + \int_{\mathbb{R}} p_n^{(1)}(y) \left(\int_{\mathbb{R}} p_m^{(2)}(z) e^V \frac{\partial V}{\partial y} dz \right) dy \end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell} (Q_1)_{n\ell} \int \int_{\mathbb{R}^2} p_{\ell}^{(1)}(y) p_m^{(2)}(z) e^{V(y,z)} dy dz \\
&\quad + \int \int_{\mathbb{R}^2} p_n^{(1)}(y) p_m^{(2)}(z) \sum_{\substack{i \geq 0 \\ j \geq 0}} i c_{ij} y^{i-1} z^j e^V dy dz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \int \int_{\mathbb{R}^2} (y^{i-1} p_n^{(1)}(y)) (z^j p_m^{(2)}(z)) dy dz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \int \int_{\mathbb{R}^2} \left(\sum_{\alpha} (L_1^{i-1})_{n\alpha} p_{\alpha}^{(1)}(y) \right) \left(\sum_{\beta} (\bar{L}_2^j)_{m\beta} p_{\beta}^{(2)}(z) \right) e^V dy dz \\
&= (Q_1)_{nm} h_m + \sum_{i,j} i c_{ij} \sum_{\alpha,\beta} (L_1^{i-1})_{n\alpha} (\bar{L}_2^j)_{m\beta} \int \int_{\mathbb{R}^2} p_{\alpha}^{(1)}(y) p_{\beta}^{(2)}(z) e^V dy dz \\
&= (Q_1)_{nm} h_m + \sum_{i,j,\alpha \geq 0} i c_{ij} (L_1^{i-1})_{n\alpha} (\bar{L}_2^j)_{m\alpha} h_{\alpha} \\
&\stackrel{(*)}{=} (Q_1)_{nm} h_m + \sum_{i,j,\alpha \geq 0} i c_{ij} (L_1^{i-1})_{n\alpha} (L_2^j)_{\alpha m} h_m \\
&= \left((Q_1)_{nm} + \sum_{i,j \geq 0} i c_{ij} (L_1^{i-1} L_2^j)_{nm} \right) h_m \\
&= \left(Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) \right)_{nm} h_m,
\end{aligned}$$

upon using, in the equality $\stackrel{(*)}{=}$, the property $h(\bar{L}_2^j)^{\top} = L_2^j h$ (see (4.4, iii)); since $h_m \neq 0$, relation (4.6, i) holds.

The proof of (4.6, ii) is similar to the previous proof, upon interchanging $1 \longleftrightarrow 2$ and $m \longleftrightarrow n$; indeed, compute for all m and n ,

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \frac{\partial}{\partial z} e^{V_2(z)} p_m^{(2)}(z) \left(\int_{\mathbb{R}} p_n^{(1)}(y) e^{V(y,z)-V_2(z)} dy \right) dz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j \geq 0} j c_{ij} \int \int_{\mathbb{R}^2} (z^{j-1} p_m^{(2)}(z)) (y^i p_n^{(1)}(y)) e^V dy dz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j} j c_{ij} \sum_{\alpha,\beta} (\bar{L}_2^{j-1})_{m\alpha} (L_1^i)_{n\beta} \int \int p_{\alpha}^2(z) p_{\beta}^1(y) e^V dy dz \\
&= (\bar{Q}_2)_{mn} h_n + \sum_{i,j,\alpha \geq 0} j c_{ij} (\bar{L}_2^{j-1})_{m\alpha} (L_1^i)_{n\alpha} h_{\alpha} \\
&\stackrel{(*)}{=} \left((\bar{Q}_2)_{mn} + \sum_{i,j,\alpha \geq 0} j c_{ij} (\bar{L}_2^{j-1})_{m\alpha} (\bar{L}_1^i)_{\alpha n} \right) h_n
\end{aligned}$$

$$= \left(\bar{Q}_2 + \left(\frac{\partial V}{\partial z}(\bar{L}_1^\top, \bar{L}_2^\top) \right)^\top \right)_{m,n} h_n,$$

upon using in $\stackrel{(*)}{=}$ that $h(\bar{L}_1^i)^\top = L_1^i h$ (see (4.5, iii)).

Now observe from (4.6)(i) and (4.11) that

$$\begin{aligned} 0 = Q_1 + \frac{\partial V}{\partial y}(L_1, L_2) &= Q_1 + \frac{\partial V_1}{\partial y}(L_1) + \frac{\partial V_{12}}{\partial y}(L_1, L_2) \\ &= M_1 + \frac{\partial V_{12}}{\partial y}(L_1, L_2), \end{aligned}$$

yielding (4.7)(i). To prove (4.7)(ii), compute from (4.6)(ii) and (4.11) that

$$\begin{aligned} 0 &= h^{-1} \bar{Q}_2 h + h^{-1} \left(\frac{\partial V}{\partial z}(\bar{L}_1^\top, \bar{L}_2^\top) \right)^\top h \\ &= h^{-1} \bar{Q}_2 h + \left(\frac{\partial V}{\partial z}(h \bar{L}_1^\top h^{-1}, h \bar{L}_2^\top h^{-1}) \right)^\top \\ &= h^{-1} \bar{Q}_2 h + \left(\frac{\partial V}{\partial z}(L_1, L_2) \right)^\top, \\ &= h^{-1} \bar{Q}_2 h + \frac{\partial V_2}{\partial z}(L_2^\top) + \left(\frac{\partial V_{12}}{\partial z}(L_1, L_2) \right)^\top \\ &= M_2^* + \left(\frac{\partial V_{12}}{\partial z}(L_1, L_2) \right)^\top, \end{aligned}$$

concluding the proof of Theorem 4.1.

Proof of Corollary 4.1.1: From (4.6(i)) and (4.6(ii)), it follows that

$$Q_1 + \sum_1^{\ell_1} i t_i L_1^{i-1} + c L_2 = 0 \quad \text{and} \quad h \bar{Q}_2^\top h^{-1} + c L_1 - \sum_1^{\ell_2} i s_i L_2^{i-1} = 0.$$

But Q_1 is zero on and above the diagonal; $\sum_1^{\ell_1} i t_i L_1^{i-1}$ has $\ell_1 - 1$ subdiagonals above the main diagonal and zeroes beyond, and thus also L_2 . Similarly \bar{Q}_2^\top has zeroes on and below the diagonal; $\sum_1^{\ell_2} i s_i L_2^{i-1}$ has $\ell_2 - 1$ subdiagonals below the main diagonal and zeroes beyond, and thus also L_1 , establishing the corollary.

Lemma 4.5. *The wave operators $S = (S_1, S_2)$ satisfy for $n = 1, 2, \dots$, and $\alpha, \beta = 0, 1, \dots$,*

$$(4.12) \quad \begin{aligned} \frac{\partial S_1}{\partial t_n} &= -(L_1^n)_\ell S_1, & \frac{\partial S_1}{\partial s_n} &= (L_2^n)_\ell S_1, & \frac{\partial S_1}{\partial c_{\alpha\beta}} &= -(L_1^\alpha L_2^\beta)_\ell S_1 \\ \frac{\partial S_2}{\partial t_n} &= (L_1^n)_u S_2, & \frac{\partial S_2}{\partial s_n} &= -(L_2^n)_u S_2, & \frac{\partial S_2}{\partial c_{\alpha\beta}} &= (L_1^\alpha L_2^\beta)_u S_2 \end{aligned}$$

and the wave vectors Ψ_1 and Ψ_2^*

$$(4.13) \quad \begin{cases} (i) & \frac{\partial(\Psi_1, \Psi_2^*)}{\partial t_n} = \left((L_1^n)_u, -((L_1^n)_u)^\top \right) (\Psi_1, \Psi_2^*) \quad n = 1, 2, \dots \\ (ii) & \frac{\partial(\Psi_1, \Psi_2^*)}{\partial s_n} = \left((L_2^n)_\ell, -((L_2^n)_\ell)^\top \right) (\Psi_1, \Psi_2^*) \\ (iii) & \frac{\partial(\Psi_1, \Psi_2^*)}{\partial c_{\alpha\beta}} = \left(-(L_1^\alpha L_2^\beta)_\ell, -((L_1^\alpha L_2^\beta)_u)^\top \right) (\Psi_1, \Psi_2^*) \quad \alpha, \beta \geq 0. \end{cases}$$

Proof: Since $p_k^{(1)}$ and $p_k^{(2)}$ are monic polynomials,

$$(4.14) \quad \frac{\partial p_k^{(i)}}{\partial c_{\alpha\beta}} = \sum_{m < k} A_{km}^{(i)} p_m^{(i)} \quad i = 1, 2$$

with $A^{(1)}$ and $A^{(2)} \in \mathcal{D}_{-\infty, -1}$. So, for arbitrary ℓ and $k \geq 0$,

$$\begin{aligned} & \frac{\partial}{\partial c_{ij}} \langle p_k^{(1)}, p_\ell^{(2)} \rangle \\ &= \left\langle \frac{\partial p_k^{(1)}}{\partial c_{ij}}, p_\ell^{(2)} \right\rangle + \left\langle p_k^{(1)}, \frac{\partial p_\ell^{(2)}}{\partial c_{ij}} \right\rangle + \langle y^i p_k^{(1)}, z^j p_\ell^{(2)} \rangle \\ &= \sum_{m < k} A_{km}^{(1)} \langle p_m^{(1)}, p_\ell^{(2)} \rangle + \sum_{m < \ell} A_{\ell m}^{(2)} \langle p_k^{(1)}, p_m^{(2)} \rangle + \sum_{m, n} (L_1^i)_{km} (\bar{L}_2^j)_{\ell n} \langle p_m^{(1)}, p_n^{(2)} \rangle \\ &= A_{k\ell}^{(1)} h_\ell + A_{\ell k}^{(2)} h_k + \sum_r (L_1^i)_{kr} (\bar{L}_2^j)_{\ell r} h_r \\ \text{or } \begin{cases} = A_{k\ell}^{(1)} h_\ell + A_{\ell k}^{(2)} h_k + \sum_r (L_1^i)_{kr} h_\ell (L_2^j)_{r\ell} \\ = A_{k\ell}^{(1)} h_\ell + A_{\ell k}^{(2)} h_k + \sum_r (\bar{L}_2^j)_{\ell r} (\bar{L}_1^i)_{rk} h_k \end{cases} & \text{using (4.4) (iii)} \end{aligned}$$

leading to

$$(4.15) \quad \begin{cases} \ell < k & 0 = \frac{\partial}{\partial c_{ij}} \langle p_k^{(1)}, p_\ell^{(2)} \rangle = (A_{k\ell}^{(1)} + (L_1^i L_2^j)_{k\ell}) h_\ell \\ \ell = k & \frac{\partial}{\partial c_{ij}} h = (L_1^i L_2^j)_0 h \\ \ell > k & 0 = \frac{\partial}{\partial c_{ij}} \langle p_k^{(1)}, p_\ell^{(2)} \rangle = (A_{\ell k}^{(2)} + (\bar{L}_2^j \bar{L}_1^i)_{\ell k}) h_k. \end{cases}$$

Setting these expressions for the matrices $A^{(i)} \in \mathcal{D}_{-\infty, -1}$ in (4.14), leads to

$$(4.16) \quad \frac{\partial p^{(1)}}{\partial c_{\alpha\beta}} = -(L_1^\alpha L_2^\beta)_\ell p^{(1)} \quad \text{and} \quad \frac{\partial p^{(2)}}{\partial c_{\alpha\beta}} = -(\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell p^{(2)}.$$

Since $p^{(1)} = S_1 \bar{\chi}$ and $p^{(2)} = \bar{S}_2 \bar{\chi}$, (4.16) leads to

$$(4.17) \quad \frac{\partial S_1}{\partial c_{\alpha\beta}} = -(L_1^\alpha L_2^\beta)_\ell S_1 \quad \text{and} \quad \frac{\partial \bar{S}_2}{\partial c_{\alpha\beta}} = -(\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell \bar{S}_2,$$

for integer $\alpha, \beta \geq 0$, and since $S_2 = h(\bar{S}_2^{-1})^\top$,

$$\begin{aligned} \frac{\partial S_2}{\partial c_{\alpha\beta}} &= \frac{\partial h}{\partial c_{\alpha\beta}} (\bar{S}_2^{-1})^\top - h \left(\bar{S}_2^{-1} \frac{\partial \bar{S}_2}{\partial c_{\alpha\beta}} \bar{S}_2^{-1} \right)^\top \\ &= \left((L_1^\alpha L_2^\beta)_0 + h((\bar{L}_2^\beta \bar{L}_1^\alpha)_\ell h^{-1}) \right) S_2, \quad \text{using (4.15), (4.17),} \\ &= \left((L_1^\alpha L_2^\beta)_0 + \left(((L_1^\alpha L_2^\beta)^\top)_\ell \right)^\top \right) S_2, \quad \text{using (4.4(iii))} \\ &= (L_1^\alpha L_2^\beta)_u S_2, \end{aligned}$$

from which it follows that

$$(4.18) \quad \frac{\partial (S_2^{-1})^\top}{\partial c_{\alpha\beta}} = -((L_1^\alpha L_2^\beta)_u)^\top (S_2^{-1})^\top.$$

In particular, setting $c_{n0} = t_n$ or $-c_{0n} = s_n$ leads to (4.12).

Since, according to the definition (4.3),

$$\Psi_1 = e^{\Sigma t_k z^k} S_1 \bar{\chi}(z) \quad \text{and} \quad \Psi_2^* = e^{-\Sigma s_k z^{-k}} (S_2^{-1})^\top \bar{\chi}(z^{-1})$$

depends on $c_{\alpha\beta}$, $\alpha, \beta \geq 1$, through S_1 and S_2 only, the expressions for $\partial S_1 / \partial c_{\alpha\beta}$ and $\partial S_2 / \partial c_{\alpha\beta}$ yield the corresponding derivatives (4.13, iii). Equations (4.13, i) and (4.13, ii) follow from (4.13, iii), taking into account the fact that t_n and s_n also appear in the exponents of (Ψ_1, Ψ_2^*) . This ends the proof of Lemma 4.5.

Proof of Theorem 4.2: The formulas

$$L_1 = S_1 \Lambda S_1^{-1}, \quad L_2 = S_2 \Lambda^\top S_2^{-1},$$

and the expression (4.8) for m_∞ combined with the differential equations (4.12) for S_i lead to

$$\begin{aligned} \frac{\partial m_\infty}{\partial c_{\alpha\beta}} &= \frac{\partial S_1^{-1} S_2}{\partial c_{\alpha\beta}} \\ &= S_1^{-1} (L_1^\alpha L_2^\beta)_\ell S_1^{-1} S_1 S_2 + S_1^{-1} (L_1^\alpha L_2^\beta)_u S_2 \\ &= S_1^{-1} L_1^\alpha L_2^\beta S_2 \\ &= \Lambda^\alpha S_1^{-1} S_2 \Lambda^{\top\beta} = \Lambda^\alpha m_\infty \Lambda^{\top\beta}, \end{aligned}$$

and thus the conclusion of Theorem 4.2. Corollary 4.2.1 is an immediate consequence of Theorem 4.2, while Corollary 4.2.2 follows from (1.16), (1.17) and (4.3).

5 Virasoro constraints on two-matrix integrals

For the general 2-Toda lattice, define the bi-infinite matrices for all integer $i \geq -1$,

$$(5.1) \quad v_i^{(1)} = L_1^{i+1} (M_1 + \frac{\partial V_{12}}{\partial y} (L_1, L_2)), \quad v_i^{(2)} = -(M_2^{*\top} + \frac{\partial V_{12}}{\partial z} (L_1, L_2)) L_2^{i+1},$$

(bi-infinite 2-Toda lattice).

Note that, according to the formula preceeding (1.10), we have

$$(5.2) \quad M_2^{*\top} = L_2^{-1} - M_2.$$

In view of formulas (1.39), define differential operators $\mathcal{K}_{m,i}^{(\alpha)}$ and $\mathcal{L}_{m,i}^{(\alpha)}$ for $m \geq 0$, $i \geq -1$, $\alpha = 1, 2$,

$$\begin{aligned} (5.3) \quad \mathcal{K}_{m,i}^{(1)} &:= \frac{1}{2} W_{m,i}^{(2)} + (i+1) W_{n,i}^{(1)} + \sum_{r,s \geq 1} r c_{rs} \frac{\partial}{\partial c_{i+r,s}} \\ &= \frac{1}{2} J_i^{(2)} + (m + \frac{i+1}{2}) J_i^{(1)} + \sum_{r,s \geq 1} r c_{rs} \frac{\partial}{\partial c_{i+r,s}} + \frac{m(m+1)}{2} \delta_{i0} \\ &=: \mathcal{L}_{m,i}^{(1)} + \frac{m(m+1)}{2} \delta_{i0} \end{aligned}$$

and

(5.4)

$$\begin{aligned}
\mathcal{K}_{m,i}^{(2)} &:= \frac{1}{2} \tilde{W}_{m-1,i}^{(2)} - \tilde{W}_{m-1,i}^{(1)} + \sum_{r,s \geq 1} s c_{rs} \frac{\partial}{\partial c_{r,s+i}} \\
&= \frac{1}{2} \tilde{J}_i^{(2)} - (m + \frac{i+1}{2}) \tilde{J}_i^{(1)} + \sum_{r,s \geq 1} s c_{rs} \frac{\partial}{\partial c_{r,s+i}} + \frac{(m-1)(m+2)}{2} \delta_{i0} \\
&=: \mathcal{L}_{m,i}^{(2)} + \frac{(m-1)(m+2)}{2} \delta_{i0}.
\end{aligned}$$

Consider the usual weight e^V , the corresponding string-orthogonal polynomials, the semi-infinite wave vectors Ψ_1 and Ψ_2^* derived from them in (4.3) and the semi-infinite matrices L_1, L_2, M_1, M_2^* defined in (4.4) and (4.5). Note that in this context neither L_2^{-1} nor M_2 appearing in (5.2) make sense although the combination $M_2^{*\top} = L_2^{-1} - M_2$ makes perfectly good sense.

Recall the bracket $\{, \}$ introduced in (2.5) and made explicit in (2.6); we now state

Theorem 5.1. *For the general 2-Toda lattice, the bi-infinite matrices $v_i^{(\alpha)}$ ($i \geq -1$, $\alpha = 1, 2$) form a (decoupled) algebra $Vir^+ \otimes Vir^+$ for the bracket $\{, \}$:*

$$(5.5) \quad \{v_i^{(\alpha)}, v_j^{(\alpha)}\} = (-1)^{\alpha+1} (i-j) v_{i+j}^{(\alpha)} \quad \{v_i^{(1)}, v_j^{(2)}\} = 0.$$

For string-orthogonal polynomials, we have

$$v_i^{(\alpha)} = 0 \quad \text{and} \quad \tau_0 = 1;$$

they imply for

$$\tau_m = \frac{1}{m!} \int \int_{u,v \in \mathbb{R}^m} e^{\sum_1^m V(u_k, v_k)} \prod_{i < j} (u_i - u_j) \prod_{i < j} (v_i - v_j) du dv$$

the algebra $Vir^+ \otimes Vir^+$ of constraints

$$(5.6) \quad \left(\mathcal{L}_{m,i}^{(\alpha)} + \frac{m(m+1)}{2} \delta_{i,0} \right) \tau_m = 0, \quad \text{for } m \geq 0, i \geq -1, \alpha = 1, 2.$$

Lemma 5.2. *For the general 2-Toda lattice, we have the following correspondence for the vector field $\mathbf{Y}_{v_i^{(\alpha)}}$ acting on the wave functions (Ψ_1, Ψ_2^*) :*

$$\begin{aligned}
(5.7) \quad -\frac{((v_i^{(\alpha)})_\ell \Psi_1)_m}{\Psi_{1,m} z^m} &= (e^{-\eta} - 1) \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} \\
-\frac{((v_i^{(\alpha)})_u^\top \Psi_2^*)_m}{\psi_{2,m}^* z^{-m}} &= (e^{\tilde{\eta}} - 1) \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} - \frac{\mathcal{L}_{m+1,i}^{(\alpha)} \tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} - (m+1) \delta_{i0}
\end{aligned}$$

for $i, m \in \mathbf{Z}, \alpha = 1, 2$.

Proof of Lemma 5.2: At first observe that $v_i^{(1)}$ and $v_i^{(2)}$ defined in (5.1) equals,

$$\begin{aligned} v_i^{(1)} &= M_1 L_1^{i+1} + (i+1)L_1^i + \sum_{r,s \geq 1} r c_{rs} L_1^{i+r} L_2^s \\ v_i^{(2)} &= M_2 L_2^{i+1} - L_2^i - \sum_{r,s \geq 1} s c_{rs} L_1^r L_2^{i+s}. \end{aligned}$$

Then using (1.27) and (1.27'), (4.13,(iii)), Corollary 1.2.1 and the notations (5.3) and (5.4), we have the following:

$$\begin{aligned} \frac{(\mathbf{Y}_{v_i^{(\alpha)}} \Psi_1)_m}{\Psi_{1,m} z^m} &= (-1)^\alpha \frac{((v_i^{(\alpha)})_\ell \Psi_1)_m}{\Psi_{1,m} z^m} = (e^{-\eta} - 1) \frac{\mathcal{K}_{m,i}^{(\alpha)} \tau_m}{\tau_m} = (e^{-\eta} - 1) \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} \\ \frac{(\mathbf{Y}_{v_i^{(\alpha)}} \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} &= (-1)^\alpha \frac{((v_i^{(\alpha)})_u^\top \Psi_2^*)_m}{\Psi_{2,m}^* z^{-m}} \\ &= (e^{\tilde{\eta}} - 1) \frac{\mathcal{K}_{m,i}^{(\alpha)} \tau_m}{\tau_m} - \frac{\mathcal{K}_{m+1,i}^{(\alpha)} \tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{K}_{m,i}^{(\alpha)} \tau_m}{\tau_m}, \\ &= (e^{\tilde{\eta}} - 1) \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} - \frac{\mathcal{L}_{m+1,i}^{(\alpha)} \tau_{m+1}}{\tau_{m+1}} + \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} - (m+1)\delta_{i0}, \end{aligned}$$

where the term $(m+1)\delta_{i0}$ is caused by the constant which distinguishes $\mathcal{L}_{m,i}^{(\alpha)}$ from $\mathcal{K}_{m,i}^{(\alpha)}$, ending the proof of Lemma 5.2.

Proof of Theorem 5.1: That the matrices $v_i^{(\alpha)}$ form an algebra $Vir^+ \otimes Vir^+$ for $\{ , \}$ with structure relations (5.5) follows immediately from Theorem 2.3. Since the maps

$$v, \{ , \} \mapsto \mathbf{Y}_v, [,]$$

and

$$\mathbf{Y}_v \text{ acting on } L \mapsto \mathbf{Y}_v \text{ acting on } \tau$$

are Lie algebra homomorphisms, the vector fields $\mathbf{Y}_{v_i^{(\alpha)}}$ induce on τ an algebra $Vir^+ \otimes Vir^+$ of constraints, as well.

For string-orthogonal polynomials, the matrices L_1, M_1, L_2 and M_2^* satisfy the string relations (4.7), implying, upon multiplication by L_1^{i+1} and L_2^{i+1} respectively, that both $v_i^{(1)}$ and $v_i^{(2)} = 0$ for $i \geq -1$; thus the right hand side of (5.7) vanishes. But the terms on the right hand side of (5.7), containing $e^{-\eta} - 1$ and $e^{\tilde{\eta}} - 1$, are Taylor

series in z^{-1} and z respectively, without independent term, whereas the remaining part is independent of z ; therefore we have

$$\frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} = a^{(\alpha)}(m, i, c), \quad m \geq 1 \text{ and } \alpha = 1, 2,$$

is a function of $m, i, c = (c_{ij})$, independent of t and s . Substituting the latter into the right hand side of (5.7), yields the difference relation

$$(5.8) \quad \frac{\mathcal{L}_{m+1,i}^{(\alpha)} \tau_{m+1}}{\tau_{m+1}} - \frac{\mathcal{L}_{m,i}^{(\alpha)} \tau_m}{\tau_m} + (m+1)\delta_{i0} = a^{(\alpha)}(m+1, i, c) - a^{(\alpha)}(m, i, c) + (m+1)\delta_{i0} = 0,$$

for $m \geq 0, i \geq -1$ and $\alpha = 1, 2$. Moreover, since $\mathcal{L}_{0,i}^{(\alpha)}$ (for $i \geq -1$) is a differentiation operator, and since $\tau_0 = 1$, we have the boundary condition

$$(5.9) \quad a^{(\alpha)}(0, i, c) = \frac{\mathcal{L}_{0,i}^{(\alpha)} \tau_0}{\tau_0} = 0, \quad \text{for } i \geq -1.$$

So, the difference relation (5.8) together with the boundary condition (5.9) implies

$$a^{(\alpha)}(m, i, c) = -\frac{m(m+1)}{2} \delta_{i0}, \quad m \geq 0, i \geq -1,$$

establishing Theorem 5.1.

Corollary 5.1.1. *For the two-matrix integral τ_N , relations (5.6) are equivalent to*

$$\begin{aligned} & \left(J_i^{(2)} + (2N + i + 1)J_i^{(1)} + N(N+1)J_i^{(0)} + 2 \sum_{r,s \geq 1} r c_{rs} \sum_{\alpha=0}^{N-1} (L_1^{i+r} L_2^s)_{\alpha\alpha} \right) \tau_N = 0 \\ & \left(\tilde{J}_i^{(2)} - (2N + i + 1)\tilde{J}_i^{(1)} + N(N+1)\tilde{J}_i^{(0)} + 2 \sum_{r,s \geq 1} s c_{rs} \sum_{\alpha=0}^{N-1} (L_1^r L_2^{s+i})_{\alpha\alpha} \right) \tau_N = 0, \\ & \text{for } i \geq -1 \text{ and } N \geq 0. \end{aligned}$$

In particular, when $V_{12} = c y z$, we have

$$\left(J_i^{(2)} + (2N + i + 1)J_i^{(1)} + N(N+1)J_i^{(0)} \right) \tau_N + 2c p_{i+N}(\tilde{\partial}_t) p_N(-\tilde{\partial}_s) \tau_1 \circ \tau_{N-1} = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N + i + 1)\tilde{J}_i^{(1)} + N(N+1)\tilde{J}_i^{(0)} \right) \tau_N + 2c p_N(\tilde{\partial}_t) p_{i+N}(-\tilde{\partial}_s) \tau_1 \circ \tau_{N-1} = 0$$

or, in terms of the moments μ_{ij} ,

$$\left(J_i^{(2)} + (2N + i + 1)J_i^{(1)} + N(N + 1)J_i^{(0)} \right) \tau_N + 2c \sum_{\substack{k+k'=N+i \\ \ell+\ell'=N \\ k,k',\ell,\ell' \geq 0}} \mu_{k\ell} p_{k'}(-\tilde{\partial}_t) p_{\ell'}(\tilde{\partial}_s) \tau_{N-1} = 0$$

$$\left(\tilde{J}_i^{(2)} - (2N + i + 1)\tilde{J}_i^{(1)} + N(N + 1)\tilde{J}_i^{(0)} \right) \tau_N + 2c \sum_{\substack{k+k'=N \\ \ell+\ell'=N+i \\ k,k',\ell,\ell' \geq 0}} \mu_{k\ell} p_{k'}(-\tilde{\partial}_t) p_{\ell'}(\tilde{\partial}_s) \tau_{N-1} = 0$$

for $i \geq -1$ and $N \geq 1$.

Proof: It suffices to replace the partial derivatives¹⁵

$$\begin{aligned} \frac{\partial \tau_N}{\partial c_{\alpha\beta}} &= N! \sum_{i=0}^{N-1} \frac{\partial h_i}{\partial c_{\alpha\beta}} h_0 \dots \widehat{h_i} \dots h_{N-1}, \quad \text{using Theorem 3.2,} \\ &= N! \sum_{i=0}^{N-1} (L_1^\alpha L_2^\beta)_{ii} h_i h_0 \dots \widehat{h_i} \dots h_{N-1}, \quad \text{using Proposition 1.1,} \\ &= N! \sum_{i=0}^{N-1} (L_1^\alpha L_2^\beta)_{ii} \prod_0^{N-1} h_k \\ (*) &= \tau_N \sum_{i=0}^{N-1} (L_1^\alpha L_2^\beta)_{ii}, \quad \text{using (3.9),} \\ (**) &= p_{\alpha+N-1}(\tilde{\partial}_t) p_{\beta+N-1}(\tilde{\partial}_s) \tau_1 \circ \tau_{N-1}, \quad \text{using theorem 4.3,} \end{aligned}$$

in the expression $\mathcal{L}_{m,i}^{(\alpha)}$ (see (5.3) and (5.4)) of (5.6) by the multiplication operator (*). For $V_{12} = cyz$, one replaces $\frac{\partial \tau_N}{\partial c_{\alpha\beta}}$ by the Hirota-type expression (**) and then one sets all $c_{ij} = 0$, for all $i, j \geq 1$, except $c_{11} = c$; the second set of equations uses the formula in Remark 1, following Theorem 4.3, thus ending the proof of Corollary 5.1.1.

Note that, since

$$\left(\frac{\partial}{\partial z} \right)^p z^n = \sum_{k=0}^{p \wedge n} \binom{p}{k} (n)_k z^{n-k} \left(\frac{\partial}{\partial z} \right)^{p-k} =: \sum_{k=0}^{p \wedge n} \alpha_k^{(n,p)} z^{n-k} \left(\frac{\partial}{\partial z} \right)^{p-k}, \quad p, n \geq 0$$

¹⁵ $\widehat{h_i}$ means with h_i omitted

we have using (1.20), that

$$\begin{aligned} [L_1^n, M_1^p] &= \sum_{k=1}^{p \wedge n} \alpha_k^{(n,p)} M_1^{p-k} L_1^{n-k}, \quad \text{with } \alpha_k^{(n,p)} := \binom{p}{k} (n)_k \\ (M_2 - L_2^{-1})^n &= \sum_{k=0}^n \beta_k^{(n)} M_2^{n-k} L_2^{-k}, \quad \text{with } \beta_k^{(n)} := \binom{n}{k} (-1)^k, \quad \beta_0^{(n)} = 1. \end{aligned}$$

Defining

$$(5.10) \quad \bar{\alpha}_k^{(i,j)} := \frac{\alpha_k^{(i,j)}}{(-c)^j (j-k+1)} \quad \text{and} \quad \bar{\beta}_k^{(i)} = \frac{\beta_k^{(i)}}{(i-k+1)c^i},$$

we now state

Theorem 5.2. *For $V_{12} = cyz$, we have*

$$\left(\sum_{k=0}^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)} \right) \tau_m = \frac{i!}{(-c)^i} \delta_{ij} \tau_m, \quad i, j \geq 0.$$

Proof: For $V_{12} = cyz$, the string equations $v_{-1}^{(\alpha)} = 0$ (see (5.1)) reduce to

$$(5.11) \quad M_1 + cL_2 = 0 \quad \text{and} \quad M_2^{*\top} + cL_1 = 0,$$

implying

$$\left(-\frac{1}{c}\right)^j L_1^i M_1^j - L_1^i L_2^j = 0 \quad \text{and} \quad \left(\frac{1}{c}\right)^i (-M_2^{*\top})^i L_2^j - L_1^i L_2^j = 0.$$

Subtracting we find, upon replacing $M_2^{*\top}$ by the expression (5.2),

$$\begin{aligned} 0 &= \left(-\frac{1}{c}\right)^j L_1^i M_1^j - \left(\frac{1}{c}\right)^i (-M_2^{*\top})^i L_2^j \\ &= \left(-\frac{1}{c}\right)^j L_1^i M_1^j - \left(\frac{1}{c}\right)^i (M_2 - L_2^{-1})^i L_2^j \\ &= \sum_{k=0}^{i \wedge j} \frac{\alpha_k^{(i,j)}}{(-c)^j} M_1^{j-k} L_1^{i-k} - \sum_{k=0}^i \frac{\beta_k^{(i)}}{c^i} M_2^{i-k} L_2^{j-k}. \end{aligned}$$

Applying Corollary 1.2.1, we have, using the notation (5.10),

$$\begin{aligned} (5.12) \quad 0 &= - \frac{\left(\left(\sum_{k=0}^{i \wedge j} \frac{\alpha_k^{(i,j)}}{(-c)^j} M_1^{j-k} L_1^{i-k} - \sum_{k=0}^i \frac{\beta_k^{(i)}}{c^i} M_2^{i-k} L_2^{j-k} \right)_{\ell} \Psi_1 \right)_m}{\Psi_{1,m} z^m} \\ &= (e^{-\eta} - 1) \frac{\left(\sum_0^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)} \right) \tau_m}{\tau_m} \end{aligned}$$

and

$$\begin{aligned}
0 &= \frac{-\left(\left(\sum_{k=0}^{i \wedge j} \frac{\alpha_k^{(i,j)}}{(-c)^j} M_1^{j-k} L_1^{i-k} - \sum_{k=0}^i \frac{\beta_k^{(i)}}{c^i} M_2^{i-k} L_2^{j-k}\right)_u\right)^\top \Psi_2^*}{\Psi_{2,m}^* z^{-m}} \Big|_m \\
(5.13) \quad &= e^{\tilde{\eta}} \frac{\left(\sum_0^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)}\right) \tau_m}{\tau_m} \\
&\quad - \frac{\left(\sum_{k=0}^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{m+1,i-j}^{(j-k+1)} + \sum_{k=0}^i \bar{\beta}_k^{(i)} \tilde{W}_{m,j-i}^{(i-k+1)}\right) \tau_{m+1}}{\tau_{m+1}}.
\end{aligned}$$

Upon using the same argument on (5.12) and (5.13) as in the proof of Theorem 5.1, one shows the ratio is independent of t and s ; so

$$(5.14) \quad \frac{\left(\sum_0^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{m,i-j}^{(j-k+1)} + \sum_0^i \bar{\beta}_k^{(i)} \tilde{W}_{m-1,j-i}^{(i-k+1)}\right) \tau_m}{\tau_m} =: a^{ij}(m, c),$$

and according to (5.13),

$$a^{ij}(m+1, c) = a^{ij}(m, c), \quad m = 0, 1, 2, \dots$$

Then, we compute (5.14) for $m = 0$:

$$\begin{aligned}
a^{ij}(0, c) &= \frac{\left(\sum_{k=0}^{i \wedge j} \bar{\alpha}_k^{(i,j)} W_{0,i-j}^{(j-k+1)} + \sum_{k=0}^i \bar{\beta}_k^i \tilde{W}_{-1,j-i}^{(i-k+1)}\right) \tau_0}{\tau_0} \\
&= \frac{\left(\sum_{k=0}^{i \wedge j} \bar{\alpha}_k^{(i,j)}(0)_{j-k+1} \delta_{i,j} + \sum_{k=0}^i \bar{\beta}_k^i(1)_{i-k+1} \delta_{i,j}\right) \tau_0}{\tau_0} \\
&\quad + \frac{\left((\text{differential operator}) + (\text{polynomial in } t)\right) \tau_0}{\tau_0}, \quad \text{using (1.49),} \\
&= \bar{\beta}_i^i(1)_1 \delta_{i,j} + (\text{polynomial in } t) \quad \text{since } \tau_0 = 1, (0)_{j-k+1} = 0 \\
&= \frac{(i)_i (-1)^i}{c^i} \delta_{i,j}, \quad \text{since } a^{ij}(0, c) \text{ is independent of } t \\
&= \frac{i!}{(-c)^i} \delta_{i,j}.
\end{aligned}$$

References

- [AvM1] Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional W_p -gravity, *Comm. Math. Phys.* 147, 25-56 (1992).
- [AvM2] Adler, M., van Moerbeke, P.: Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials, *Duke Math. J.* 80, 1995.
- [AvM3] Adler, M., van Moerbeke, P.: String-orthogonal polynomials and the Borel decomposition, to appear (1996).
- [ASV1] Adler, M., Shiota, T., van Moerbeke, P.: From the w -algebra to its central extension: a tau-function approach, *Phys. Lett. A* 194, 33-43 (1994).
- [ASV2] Adler, M., Shiota, T., van Moerbeke, P.: A Lax representation for the vertex operator and the central extension, *Comm. Math. Phys.*, 171, 547-588 (1995).
- [ASV3] Adler, M., Shiota, T., van Moerbeke, P.: Random matrices, vertex operators and the Virasoro algebra, *Phys. Lett. A* 208, 67-78 (1995).
- [ASV4] Adler, M., Shiota, T., van Moerbeke, P.: The spectrum of coupled random matrices (to appear 1996).
- [AMSV] Adler, M., Morozov, A., Shiota, T., van Moerbeke, P.: Solving $[L, P] = L$ and matrix Laplace transforms, *Comm. Math. Phys.*, 1996
- [AS1] Ahn, Ch., Shigemoto, K.: Multi-matrix model and 2D Toda multicomponent hierarchy, *Phys. Letters B*, 263, 44-50 (1991).
- [AS2] Ahn, Ch., Shigemoto, K.: One point functions of loops and constraint equations of the multimatrix models at finite N , *Phys. Lett. B*, 285, 42-48 (1992).
- [AGL] Alvarez-Gaumé, L., Gomez, C., Lacki, J.: Integrability in random matrix models, *Phys. Letters B*, vol. 253, (1,2), 56-62.
- [BK] Brezin, E., Kazakov, V.A.: Exactly solvable field theories of closed strong, *Phys. Letters B* 236, 144-150 (1976).

- [CMM] Chadha, S., Mahoux, G., Mehta, M.L.: J. Phys. A14, p. 576 (1981).
- [DJKM] Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations, Proc. RIMS Symp. Nonlinear integrable systems, Classical and quantum theory (Kyoto 1981), pp. 39-119. Singapore : World Scientific 1983.
- [D1] Dickey, L.A.: Soliton equations and Hamiltonian Systems, World Scientific, 1991.
- [D2] Dickey, L.A.: Additional Symmetry of KP Grassmannian and the String Equation, Modern Phys. Lett. A, vol. 8, No 13, p. 1259-1272 (1993).
- [D3] Dickey, L.A.: On additional symmetry of the KP hierarchy and Sato's Bäcklund transformation, hep-th 9312015, 1993.
- [DS] Douglas, M., Shenker, S.: Nucl. Phys. B 335, 635 (1990)
- [Fu] Fuchssteiner, B.: Mastersymmetries, higher order time-dependent symmetries and conserved densities of non-linear evolution eqts, Progr. Th. Phys. 70, 1508-1522 (1983).
- [GMO] Gerasimov, A., Marshakov, A., Mironov, A., Morozov, A., Orlov, A. : Matrix models of Two-dimensional gravity and Toda theory, Nuclear Physics B357, 565-618 (1991).
- [G-M] Gross, D., Migdal, A. : Phys. Rev. Lett. 64, p.127 (1990).
- [Ka] Kac, V.G. : Infinite-dimensional Lie algebras, 3rd edition, Cambridge Univ. Press.
- [KR] Kac, V., Raina, A.K.: Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras, Advanced Series in Mathematical Physics, vol. 2, 1987, World Scientific.
- [M] Mehta, M.L.: A method of Integration over Matrix Variables, Comm. Math. Phys., 79, 327-340 (1981).
- [Ma1] Martinec, E.J. : On the Origin of Integrability in Matrix Models, Comm. Math. Phys.138, 437-449 (1991)

- [Mo] Morozov, A.: Integrability and matrix models. Usp. Fiz. Nauk, **164**, (1) 3–62 (1994) (Physics Uspekhi, **37** 1–55 (1994), hep-th/9303139), and references therein.
- [Moo] Moore, G. : Geometry of the String Equations, Comm. Math. Phys. 133, 261-304 (1990)
- [Mu1] Mulase, M.: Algebraic theory of the KP equation, preprint (1993).
- [Mu2] Mulase, M.: Matrix integrals and integrable systems, preprint (1994).
- [O-Sc] Orlov, A.Y., Schulman, E.I. : Additional Symmetries for Integrable and Conformal Algebra Representation, Letters in Math. Phys., 12 171-179 (1986)
- [U-T] Ueno, K., Takasaki, K. : Toda Lattice Hierarchy, Adv. Studies in Pure Math. 4, (1984) 1-95.
- [VDL1] van de Leur, J.W. : The Adler-Shiota-van Moerbeke formula (preprint 1994)
- [VDL2] van de Leur, J.W. : (preprint 1994)
- [vM2] van Moerbeke, P.: Integrable foundations of string theory, in Lectures on Integrable systems, Proceedings of the CIMPA-school, 1991, Ed.: O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach, World scientific, pp 163–267 (1994).
- [W1] Witten, Ed. : Two-dimensional gravity and intersection theory of moduli space, Harvard University lecture; May 1990, Journal of diff. geometry 1991.
- [W2] Witten, Ed. : On the Kontsevich Model and other Models of Two Dimensional Gravity, IASSNS-HEP-91/24 (6/1991) preprint (1991)